ANALYSIS OF INELASTIC FRAMES WITH STRAIN HARDENING AND LIMITED DUCTILITY

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To

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CERTIFICATE

This is to certify that the thesis entitled " ANALYSIS OF INELASTIC FRAMES WITH STRAIN HARDENING AND LIMITED DUCTILITY " by ASHOK KUMAR SINHA is a record of work carried out under my supervision and that it has not been submitted elsewhere for a degree.

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SYNOPSIS

The general problem of ultimate load analysis of structures with sections exhibiting bilinear strain hardening material property is extremely complex and highly computative. A method of ultimate load analysis of such structures, ideal for computer application, is presented. All the conditions of plastic analysis are directly satisfied without bringing in the concept of collapse mechanism. The problem is formulated as a linear programming problem with the load factor as the objective function. Equilibrium equations, yield criterion and the criterion for rotation compatibility are posed as constraint equations.

The presented algorithm consists of two cycles of operation. In the first cycle the structure is analysed with the assumption that sections have elastic-plastic characteristics. The inelastic rotation and rotation capacity of each section are computed and then additional constraints for rotation compatibility are imposed. In the second cycle, the analysis is done with these additional constraints.

Examples have been included to illustrate the advantages of such a procedure.

ACKNOWLEDGEMENTS

I would like to express my deep appreciation and graditude to my thesis advisor, Dr. H.C.Parmeswar, for initiating me to the work presented in this thesis. But for his sustained interest, guidance and encouragement to think originally, the work presented here would not have been possible.

I wish to express my appreciation to all my friends for their kind help. I am thankful to the Staff of the Computer Centre, IIT/K for providing facilities on IBM 7044 system and to Mr. K.N.Tiwary for the excellent typing.

Ashok Kumar Sinha

Ashok Kumar Sinha

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NOMENCLATURES

λ	Load factor at any state
λ_u	Load factor at full redistribution
M^{-}_{pj}	Plastic moment capacity of Section j in Negative moment
M^{+}_{pj}	Plastic moment capacity of Section j in Positive moment
M_u	Maximum moment that a section can resist
M_y	Moment at which yielding starts.
M_i	Moment at any section i
M_{ei}	Pseudo-elastic moment at section i
m_i	Residual moment at section i
P	Load acting perpendicular to axis of member
H	Horizontal load
θ_i	Correct inelastic rotations at section i
θ'_i	Incorrect inelastic rotations
θ_{pi}	Rotation capacities
ϕ	Curvature
ϕ_p	Inelastic curvature
ϕ_u	Ultimate curvature
ϕ_y	Curvature at which yielding starts
α	Ratio of ultimate moment and yield moment
B	Ratio of ϕ_u and ϕ_y.
EI	Flexural rigidity
K_{jk}	Relative stiffness of member jk

U_{jk}	Energy in member jk
U_r	Energy of whole structure
L	Unit dimension of length
l	Inelastic length
m,n,r	System parameters, has been defined as and when used.
X_i	System parameters in linear programming problem.

SIGN CONVENTION

(a) Moment producing tension on dotted side of a member is positive.

(b) Positive residual moment act in the direction of inelastic rotation.

CHAPTER I

INTRODUCTION

Most structural materials exhibit considerable reserve strength and ductility beyond elastic range. The material behaviour in post-elastic region is generally nonlinear (Fig.1.1a). Rigorous analysis based on nonlinear M-ϕ relationship is very complex and time consuming. Even the use of a bilinear moment-curvature relationship, which is the simplest mathematical model closely representing the inelastic behaviour of the material; is a highly computative process. The development of plastic theory based on elastic-perfectly plastic (Fig.1.1b) idealisation of the material behaviour has been only a matter of convenience. This assumption of elastic-plastic moment curvature relation is justified for certain grades of steel. But materials like high carbon steel, aluminium and concrete, which have a nonlinear moment-curvature relationship, can also have very limited ductility. The two factors put together may lead to large errors if the structures are analysed and designed on the basis of assumption of ideal elastic-plastic behaviour with unlimited ductility.

1.2 Definitions

According to the elementary plastic theory, as the loads on a structure are gradually increased, plastic hinges

are gradually formed at several critical sections. At the sections where plastic hinges have appeared, the hinges continue to rotate under constant moment. The other sections continue to take shares of the applied load till sufficient number of hinges from causing the structure to collapse by the formation of a mechanism[1,2,3]. This process of transfer of moments is called the "moment redistribution". Complete moment redistribution has to take place in order to be able to exhaust the post-elastic reserve strength fully. This implies that the hinges formed earlier will freely rotate to such extents as necessary to permit the later hinges to form. Such rotations of the hinges will cause large curvatures and consequently large strains. This is where large ductility plays an important part in plastic analysis. Even for frames made of highly ductile materials, it has been recognized[4,5] that such factors as shear, axial force, local buckling, residual stresses and lateral buckling may cause the moment curvature relationship to differ markedly from the idealised one on which the concept of the collapse mechanism is based. In view of the above factors, inelastic analysis and design with materials having nonlinear behaviour and limited ductility is an important problem.

1.3 Necessity of Present Work

Several attempts have been made to take into account the above factors, piecewise[6,7,8,9]. Nevertheless, the existing methods of inelastic analysis and design are subject

to the following limitations:

1. There is no comprehensive analytical method to evaluate ultimate load providing a direct solution satisfying all the conditions of plastic analysis.

2. There is no design method adoptable for design office taking into account the effect of strain hardening and effect of limited ductility.

3. The methods are largely trial and error procedures.

4. The existing methods cannot be easily adopted to computers.

5. There is no plastic design method to handle cases of partial collapse.

1.4 Object and Scope

In the light of the above discussions, the object of the thesis is the following:

1. To propose a method of plastic analysis using linear programming where the ultimate load is evaluated without the concept of collapse mechanisms. All the conditions of plastic analysis are directly satisfied.

2. To provide a comprehensive method of analysis taking into account the effect of limited ductility and strain hardening. An algorithm is presented.

3. To provide a method for handling the case of partial collapse. In the procedure only a possible design solution is obtained as a first step in overcoming one of the difficult problem of designer.

With the availability of bigger and modern computers there is scope for the proposed method to be very effectively developed and used as a regular method for inelastic design of structures.

1.5 Proposed Method

A linear programming approach to obtain a possible design solution in cases of partial collapse is presented in Chapter V. In Chapter VI, the same approach makes use of the bilinear, strain hardening material property with limited ductility to obtain the limit load for plane frames. A two step procedure is developed to arrive at the ultimate load.

Material nonlinearity and limited ductility are taken into account by formulating constraints on inelastic rotations at critical sections in the structure. In the first step of the analysis, an elastic plastic characteristic of the section with infinite ductility is assumed. The problem is posed as a linear programming problem and the ultimate load and the corresponding moment distribution are obtained. An expression for the inelastic rotation in the above structure is developed and, finally, constraints on inelastic rotations are added to the previously formulated problem. The ultimate load of the structure is now obtained with these additional constraints.

In Appendix I, a flow chart for programming this algorithm is presented.

1.6 Assumptions

The following are the assumptions made in the method proposed.

1. The structural material may or may not be sufficiently ductile to sustain the large local strains required for complete redistribution of moments.

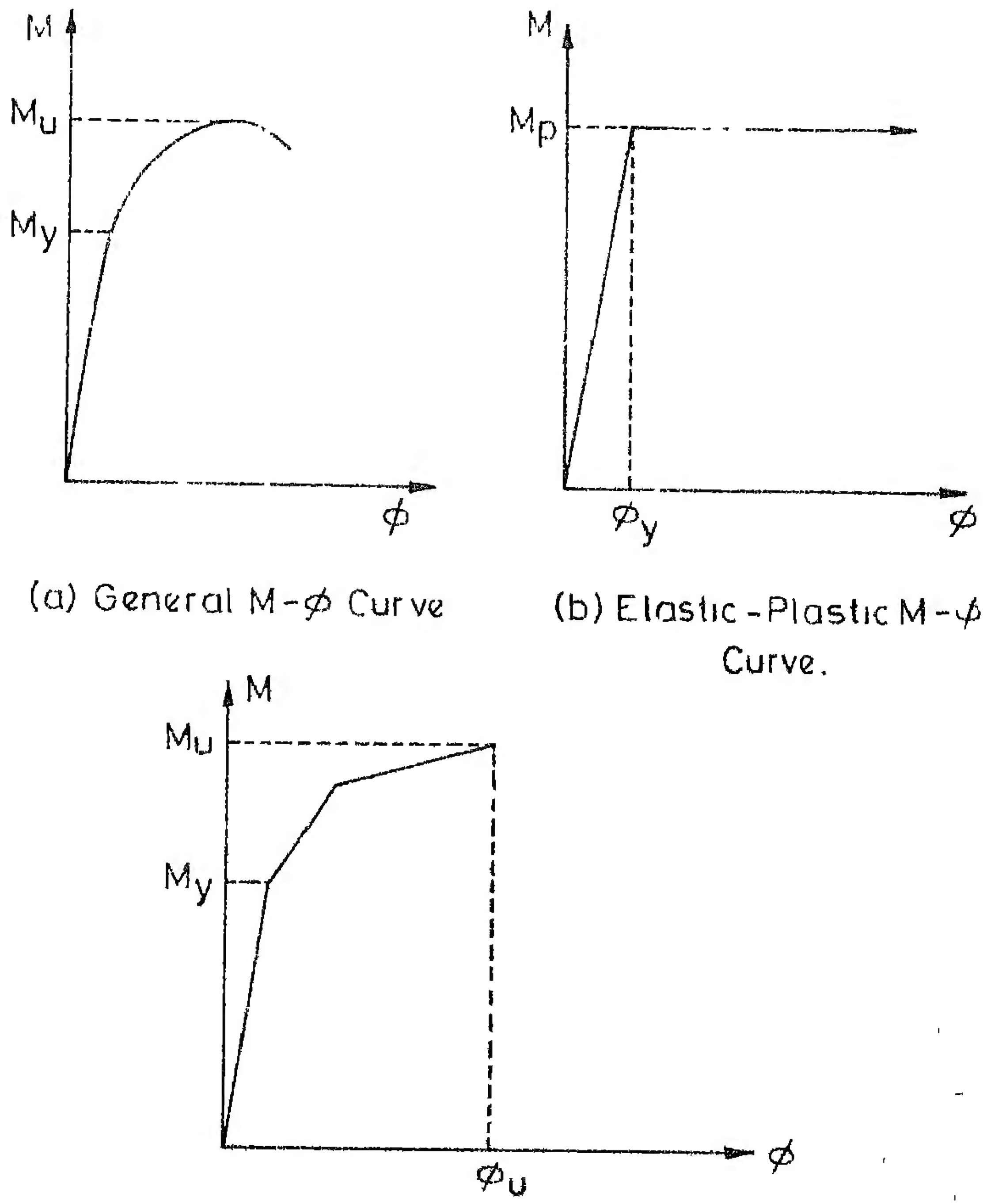
2. Because of the limited ductility and the strain hardening behaviour, inelasticity will develop around the critical sections where magnitude of moment is more than yield moment (M_y). The continuous change of slope in this inelastic zone is treated as concentrated at critical sections.

3. The effect of axial force and shear force are assumed to be included in the moment-curvature relation.

4. The structure does not become unstable until atleast one section reaches its ultimate curvature (ϕ_u).

5. The loads on structure are assumed to increase proportionately.

6. Deformations at ultimate load are small so that they do not appreciably influence the equilibrium condition for the structures.



(a) General M-ϕ Curve

(b) Elastic-Plastic M-ϕ Curve.

(c) Trilinear M-ϕ Curve.

Moment Curvature Relations.

Figure 1-1

CHAPTER II

METHODS OF PLASTIC ANALYSIS

2.1 Introduction

The elementary theory of plastic analysis or design is based on the satisfaction of three basic conditions.

1. Equilibrium
2. Yield condition
3. Mechanism condition

While the first of the above conditions ensures the structure to be in equilibrium at all stages of loading upto failure, the second ensures that the moment at any section is less than or equal to the plastic moment capacity of the section. Under the idealisation of elastic perfectly plastic moment curvature relation, a section behaves as a perfect hinge under constant moment as soon as the section reaches its plastic moment capacity. When sufficient number of sections have developed plastic hinges rendering the structure an unstable mechanism, the structure is said to have reached the mechanism condition.

Within the usual assumptions of plastic analysis, an analysis or design solution is unique if all the above three conditions are rigorously satisfied. The fact that such direct solutionsare complex in many cases has led to the following theorems[10]:

1. Lower Bound Theorem: If an equilibrium distribution of stress can be found which balances the applied load and is

everywhere less than or equal to yield stress, the structure will not fail. At most it will have just reached the ultimate load.

2. Upper Bound Theorem: For any structure, with compatible pattern of plastic deformations, the structure cannot be in equilibrium if the rate at which the external forces do work exceeds the rate of internal work dissipation.

2.2 Plastic Analysis Methods

There are several methods existing for plastic analysis and design of structures. Of these, the following are a few common ones bearing the essential features involved in most of them. The essential difference between these methods basically lies in the manner of satisfaction of the three conditions listed earlier.

1. Yield Surface: All possible mechanisms are considered and the corresponding limit equilibrium conditions are expressed as inequalities in terms of the 'n' independent loads involved. Such a set of inequalities define an n dimensional surface in an 'n' dimensional load space. A state of loading corresponding to any point within this yield surface defines a safe loading state. The load corresponding to a point on yield surface represents a state of collapse while the load conditions corresponding to points outside the surface cannot be reached. When 'n' is two, a geometric interpretation can be given to the

problem and hence a solution can be obtained geometrically. Otherwise one has to resort to mathematical programming.

Example 2.1:

Consider the frame shown in Fig.2.1(a). There are two loads P and H. The plastic moment capacities are assumed to be constant throughout the frame and equal both in positive and negative directions ($\pm M_p$).

Since the frame in question is a simple one, it is easy to see that there are three possible mechanisms. These are the beam, the panel and the combined mechanisms as shown in Fig.2.1(b),(c) and (d) respectively. Using the principle of virtual work[4] the following inequalities are obtained

for the beam mechanism	$H \leqslant 4M_p/L$
for the panel mechanism	$P \leqslant 4M_p/L$
for the combined mechanism	$P+H \leqslant 6M_p/L$

Each of these limit equilibrium conditions being linear is represented by a straight line in Fig.2.1(e). The shaded area OABCD defines the safe region for independent variation of the two loads. If P/H has a fixed given ratio μ, as in case of proportional loading, then the collapse load is given by point N where $P = \mu H$ intersect the boundary of safe region OABCD.

2. Method of General Hinge Rotation: Plastic hinges are assumed at several critical sections rendering the structure a collapse mechanism. The choice of location of such hinges

is dependent upon the loading condition and also the experience of the designer. The following relation is established between the number of hinge locations, the number of indeterminacies and the number of independent rotations:

$$m = n-r$$

where m = number of independent rotations

n = number of hinge sections

r = number of indeterminacies rotations.

The rotations at all the hinges are expressed in terms of any of these m independent hinge rotation and the general load factor is expressed using virtual work equation. These independent rotations are given independent variations until the least load factor is attained.

Example 2.2

The frame shown in Fig.2.1(a) has three redundancy and five critical sections and hence the number of independent rotations are

$$m = 5-3 = 2$$

these two independent rotations are shown in Fig.2.2(a). The virtual work equation is expressed as

$$(H\theta_1 + P\theta_2)\, L = M_p(|2\theta_1| + |\theta_2-\theta_1| + |2\,\theta_2| + |\theta_2+\theta_1|)$$

for $\frac{P}{H} = 0.75$, the ultimate load P is given by

$$\frac{PL}{M_p} = \frac{2|\theta_1| + |\theta_2-\theta_1| + |2\,\theta_2| + |\theta_1+\theta_2|}{0.75\,\theta_1 + \theta_2}$$

Arbitrary values of θ_1 and θ_2 are assumed and the corresponding value of PL/M_p is calculated. The result is shown graphically in Fig.2.2(b). The minimum value of P is $3.43M_p/L$ and occurs when $\theta_2/\theta_1 = 1$ i.e. $\theta_2 = \theta_1$.

3. Method of Combination of Mechanisms· For complex frames with high degree of redundancy, it is not possible to consider all the possible mechanisms one by one, primarily because it is difficult to conceive of all the possible mechanisms.

The method of combining mechanisms is an approximate procedure presented by Neal and Symonds[11,12]. In this method, a systematic search is made for mechanism which leads to the least value of the load factor λ. In order to ensure that the search at any stage is complete and that there is no other mechanism leading to a smaller value of λ, it is essential to compute the bending moment at all the critical sections to check the yield condition.

There are, basically three types of mechanisms that are considered for plane frames:

(a) Beam mechanism: In this mode hinges are located only in the beams. Fig.2.3(a).

(b) Panel mechanism: In this mode the hinges are located only in columns. Fig.2.3(b) and (c).

(c) Joint mechanisms: In this mode hinges are located only around a joint in all the connecting members. Fig.2.3(d).

In a given structure there are as many basic beam mechanisms as there are beams and as many panel mechanisms as there are independent panels. An independent panel is one which can deflect in one unit. Fig.2.3(b) and (c) show some typical independent panel mechanisms. There are as many joint mechanisms as there are joints. A joint is defined as one where two or more members meet. However, a certain amount of discretion in considering joint mechanisms can be used.

The total number of independent mechanisms 'm' is equal to number of equilibrium equations 'e' which is given by

$$m = e = n-r$$

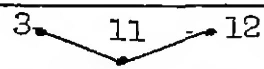
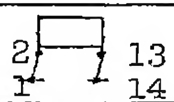
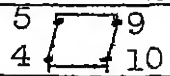
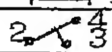
where n is number of critical sections and r is number of indeterminacy.

Example 2.3

The method of combining mechanism is illustrated by means of the frame shown in Fig.2.4(a).

The frame has 6 indeterminacies. The critical sections are numbered as 1 to 14. Therefore number of independent mechanisms are equal to eight. Of these two are beam mechanisms, Fig.2.4(b) and (c); and two are panel mechanisms, Fig.2.4(d) and (e); and four joint mechanisms, Fig.2.4(f), (g), (h) and (i). The mechanisms are combined in a systematic way as shown in Table I.

TABLE 2.1 Combination of Mechanisms for Example 2.3.

Type of mechanism	Section no. / Mechanism	1	2	3	4	5	6	7	8	9	10	11	12	13	14	λ	Extn. work done	Intn. work done
	M_p →	1	1	1	1	1	1	1	1	1	1	1	1	1	1		PLθ	$M_P\theta$
	b			-1								2	-1			4	1	4
	c						-1	2	-1							4	1	4
	d	-1	1											-1	1	5	0.8	4
	e				-1	1				-1	1					8	0.5	4
	f		-1	1	1													
	g					-1	1											
	h								-1	1								
	i										-1		-1	1				
(k)	b+d+e +f+g+h +i	-1					1		-1			2	-2		1	3.5	2.3	8
(m)	k+c	-1						2	-2			2	-2		1	3.03	3.3	10

2.3 Limitations of Plastic Analysis Methods

In addition to the methods discussed above there are several other plastic analysis and design methods[5] viz. method of inequality, method of plastic moment distribution etc. The following general observations are made about all these methods

1. All the methods are basically trial and error methods. Of the three conditions they have to satisfy (viz. equilibrium, yield and mechanism), they uniquely satisfy any two and search for a solution among these which satisfy the third condition.

2. The methods of moment distribution and method of combination of mechanism are most commonly used. Even in these methods after a particular load factor or a particular design moment set has been obtained, one has to make a complete analysis for the moments at all the critical sections; other than the hinge ones, to ensure satisfaction of yield condition.

3. For large frames one has to resort to computers in all these methods even though they are not ideally formulated for computer applications.

4. In case of partial collapse the designer is faced with the problem of performing another elastic analysis of that part of the structure which is still statically indeterminate.

5. All these methods are based on an ideal elastic perfectly plastic moment curvature relation with unlimited ductility. Since this assumption is not always true[4,5], it is more rational to restate the problem with a constraint on ductility.

6. The merit of plastic methods of design basically lies in the reserve strength of the structure existing between the first yield and collapse. Ignoring the effect of strain hardening this reserve strength is normally of the order of 15 to 20%. Since the plastic deformations are generally localised, the strain hardening region is reached at small deflections. Consequently the section tends to develop moments much greater than the idealised plastic moment capacity. Fig.2.6 shows a typical load deformation behaviour of a simple beam test reported by Driscoll and Beedle[13]. In this case a reserve strength of nearly 22% above the theoretical collapse load ignoring strain hardening is available. Therefore it is often irrational to ignore the effect of strain hardening which can sometimes lead to a reserve strength of 22% over and above the 15 to 20% reserve strength due to moment redistribution under the elastic perfectly plastic idealisation alone.

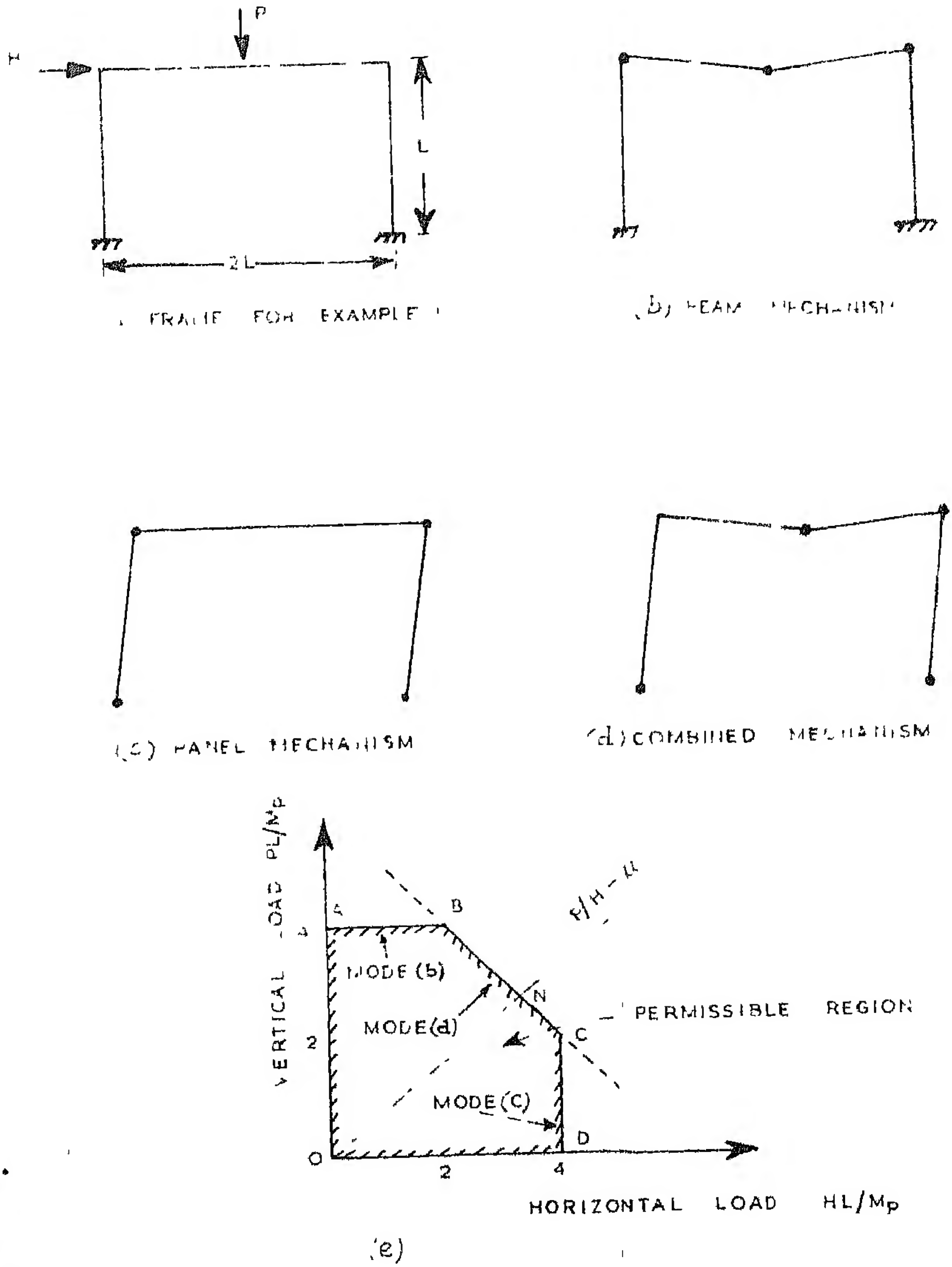
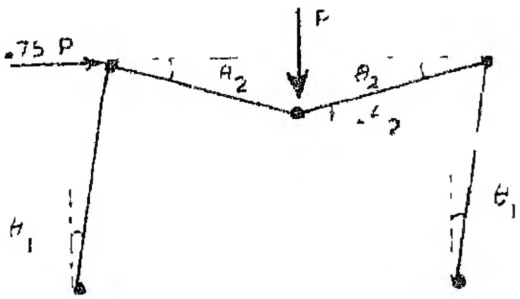
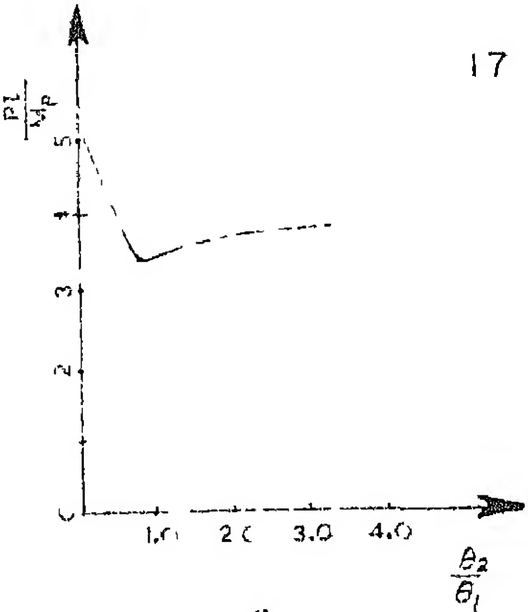

P
H
L
2L
EXAMPLE
(d) COMBINED MECHANISM
A
B
C
D
N
O
MODE (b)
MODE(d)
MODE(C)
PERMISSIBLE REGION
VERTICAL LOAD PL/Mp
HORIZONTAL LOAD HL/Mp
2
4
(e)

FIGURE 2.1



(a)

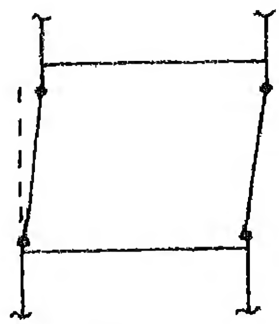


EXAMPLE 2.2

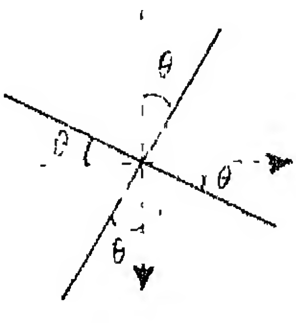
(b)

FIGURE 2.2

(a) BEAM MECHANISM

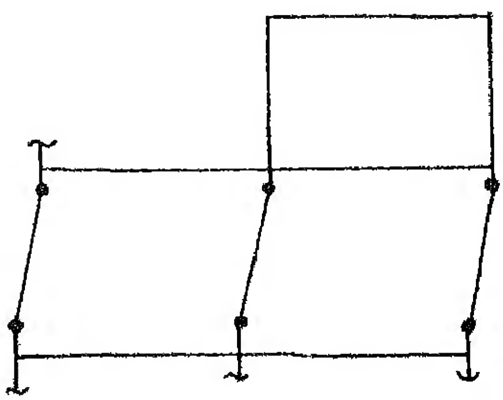


(b)



(d)

JOINT MECHANISM

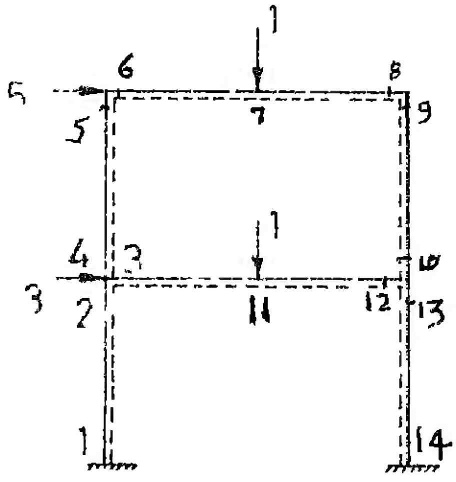


(c)

PANEL MECHANISM

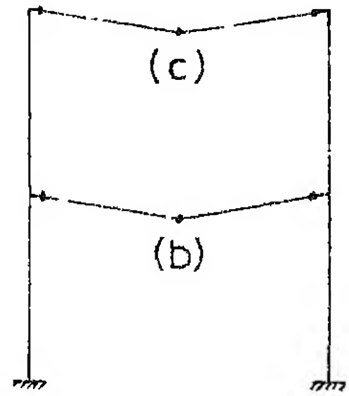
TYPES OF MECHANISMS

FIGURE 2.3

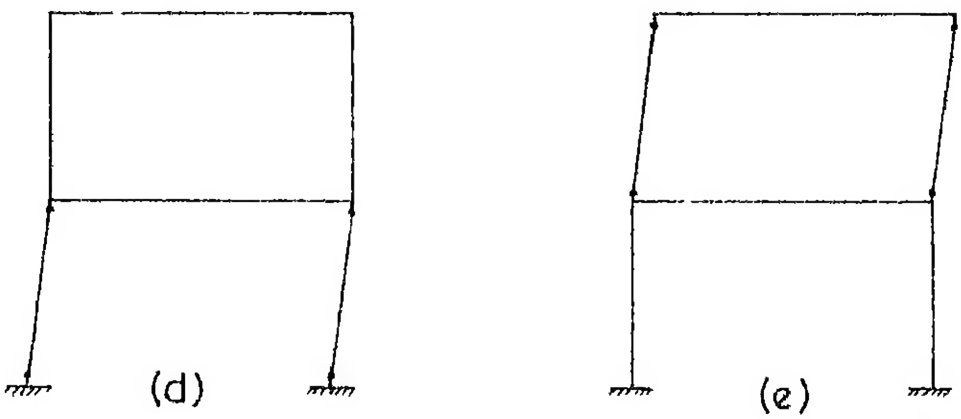


(a)

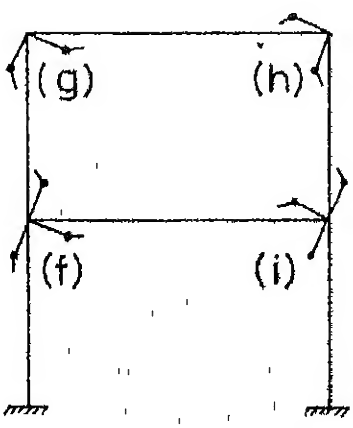
Frame for example 2 3



Beam Mechanisms



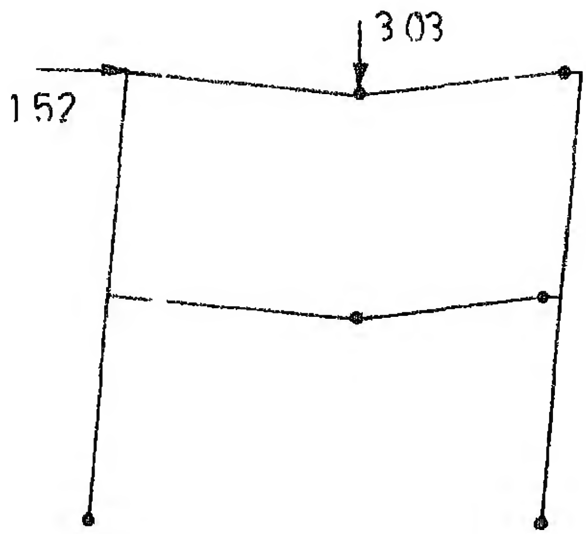
Panel Mechanisms



Joint Mechanisms.

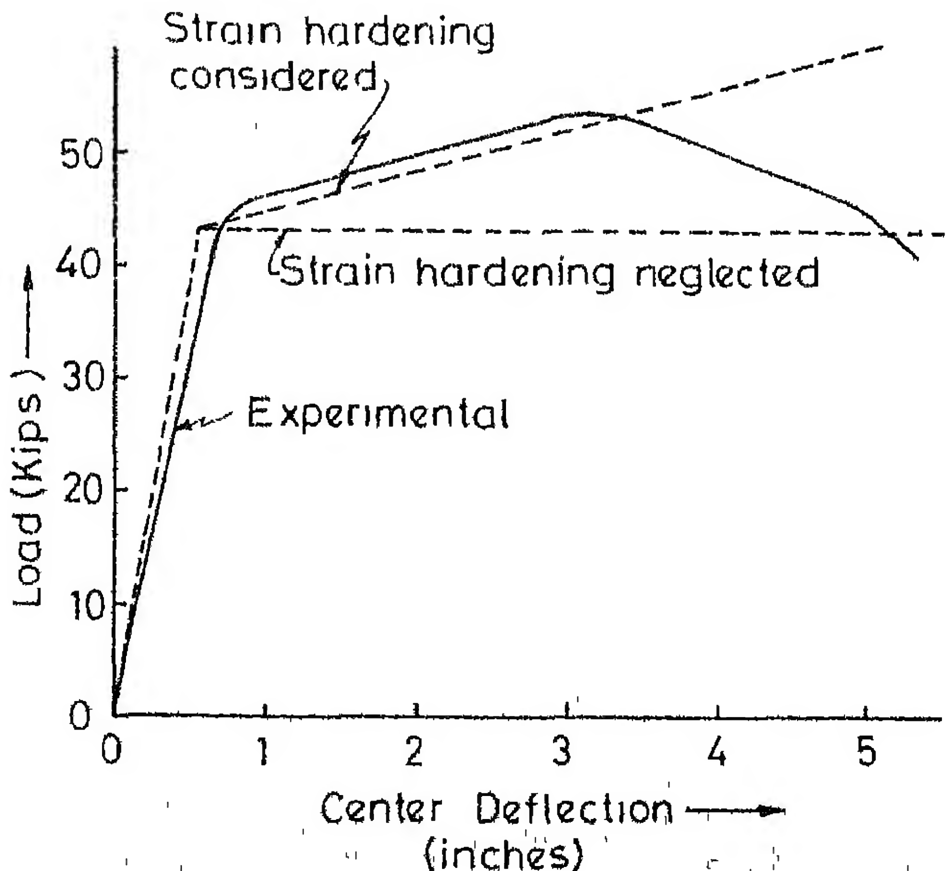
Frame for Example 2 3

Figure 2 4



Collapse Mechanism for Example 2.3

Figure 2.5



Load vs. Deflection Relationship

Figure 2.6

CHAPTER III

PROBLEM OF FRAMES WITH LIMITED DUCTILITY

3.1 Introduction

One of the main reasons for elastic perfectly plastic idealisation is that the design procedure becomes simple. Such idealisations, however, lack from one to one correspondance between the moments and the curvatures beyond elastic limit. There are several materials, whose behaviour can be idealised as above and yet limitations have to be specified on the ductility. Complications enter when such limitations have to be taken into account in design. Nonlinear relationships are presently used in Russia and similar trends are showing in other countries. Design methods based on a rigourous satisfaction of even bilinear moment-curvature, not to talk about nonlinear, are quite complex. Therefore even the bilinear methods are all approximate methods and are based on simplifying assumptions. There are several procedures proposed to specifically take this fact into account. The essential features of these methods lie in imposing the ductility limits on the rotation of the plastic hinges under gradually increasing loads. This problem is commonly referred to as the problem of "Rotational Compatibility". In these design methods the compatibility requirement is satisfied by specifying the following limitations on the plastic hinge rotations:

$$\theta_i \leq \theta_{pi} \qquad (2.1)$$

where θ_1 is the inelastic rotation at the plastic hinge section 1. This implies that the inelasticity is concentrated at the critical sections. θ_{p1} represents the rotation capacity. This implies that θ_{p1} has to be based on a more realistic M-Ø relation where a one to one correspondance between moment and curvature exists.

3.2 Limitations of the Compatibility Formulation

There are several limitations in the formulation of compatibility condition in the form $\theta_1 \leqslant \theta_{p1}$. It is essential to keep these limitations in mind while discussing the formulation. The limitations are as follows:

1. The inelastic rotation $\bar{\theta}_1$ is based on elastic perfectly plastic idealisation while θ_{p1} is based on a more accurate M-Ø relation. Therefore, satisfying the compatibility condition as $\theta_1 \leqslant \theta_{p1}$ is only an approximate and indirect way of incorporating the material behaviour into the problem. The extent of this approximation depends on the extent of deviation of the actual M-Ø model from the ideal elastic perfectly plastic one. The closer the actual M-Ø is to the ideal elastic perfectly plastic one, the lesser the significance of $\bar{\theta}_{p1}$. The farther the actual M-Ø is from the ideal elastic-perfectly plastic one, the lesser the significance of θ_1. Therefore, the formulation $\theta_1 \leqslant \theta_{p1}$ is meaningful only between these extremes.

2. By definition θ_{p1} is indirectly the area between the idealised M-Ø and the actual M-Ø. As the actual M-Ø

gets closer to the ideal one this area reduces and hence θ_{pl}. This is tending towards more severe compatibility restrictions.

3. The formulation $\theta_1 \leqslant \theta_{pl}$ loses its meaning completely when the actual M-Ø is elastic perfectly plastic but with limited curvature ϕ_u. In such cases because of the lack of one to one correspondance between the moment and the curvature beyond yielding, θ_{pl} has no meaning. Only one M-Ø model exists for both θ_1 and θ_{pl}. Yet, compatibility cannot be warranted because of limited curvature capacity.

4. Many of the existing proposals for evaluating θ_{pl} are based on parameters derived from the rotations observed in tests. These rotations need not always represent the real rotation capacity. Consider for example a simply supported beam with a concentrated load being tested. The actual M-Ø curves of the sections provided in the beam is always bounded between the extremes - an M-Ø with no ductility at all and an M-Ø which is elastic perfectly plastic with infinite ductility. In the test of the above beam, at the time of collapse, the maximum rotation that can be measured in the case of M-Ø with no ductility will be zero. In the other case when M-Ø is elastic perfectly plastic the inelastic rotation that can be measured is again zero because the beam is determinate and hence the load cannot be increased beyond the level at which the moment under the load just reaches the ultimate moment

capacity of the section. If a similar test is conducted on an indeterminate beam the maximum rotation that can be measured at any hinge section will be equal to the calculated inelastic rotation θ_1. This again is not necessarily equal to the rotation capacity because by assumption the section has infinite rotational capacity based on the elastic-perfectly plastic M-Ø with infinite ductility. In reality, however, the M-Ø relations are not elastic-perfectly plastic. But based on the above argument, it can be concluded that in case of sections whose M-Ø behaviour is very close to ideal elastic-perfectly plastic, large errors may be involved in considering the measured rotations as representing the real rotation capacities.

3.3 Inelastic Analysis and Design

The following are some of the commonly used methods for inelastic design of structures of material with limited ductility.

3.3.1 Method of Imposed Rotations

The method is based on general interpretation of the plastic behaviour given by Collometti[14]. Since then it has been established and proposed by Macchi[6,9] as a standard design method for nonlinear structures.

The most convenient method for solving inelastic problems of the nature discussed earlier is the method of imposed rotations. This method assumes inelastic rotations

as imposed at the critical sections and the statical effect of these rotations are superimposed on elastic effect of loads. This method has been practically applied to reinforced concrete and prestressed concrete. However, the method is quite general and can be applied to any material for which bilinear or trilinear M-Ø idealisation is valid. The following are the basic steps involved in the method:

1. To start with, a complete redistribution of moments is considered and the load corresponding to this moment distribution is calculated.

2. For this load, a moment diagram confirming to elastic behaviour is determined.

3. Inelastic rotation at critical sections are obtained from known trilinear moment-rotation relationship.

4. A moment distribution is obtained to cause the dislocation in the structure and this distribution is superimposed on the distribution obtained in step 2.

5. If the final moment diagram thus obtained does not violate yield condition, it will be the actual configuration at failure. Otherwise another trial with elastic moment distribution for a reduced load is required.

This method of successive approximation is restricted to rather simple continuous beams in which the degree of indeterminacy is not very high.

3.3.2 Sawyer's Method

In this method a bilinear M-ϕ relation is assumed and the rotation capacities are evaluated analytically. A bilinear M-ϕ diagram representing strain hardening is assumed to simplify the inelastic analysis and design procedures. The bilinear diagram adopted by Sawyer[8] is shown in Fig.7. For line OA, curvature is given by

$$\phi = \frac{M}{EI} \tag{3.1}$$

and for any moment M_1 between the yield moment M_y and the ultimate moment M_u, the curvature caused by inelasticity is

$$\phi_p = k_p \frac{M_1 - M_y}{EI} \tag{3.2}$$

where k_p is defined as the plasticity factor and is determined by the intercept to the left of origin (Fig.3.1).

For bilinear analysis the basic problem is the evaluation of excess in the curvature and the bending angle caused by inelasticity. For the evaluation of this inelastic bending angle, the familiar moment area principle can be used. Based on this fact Sawyer presented a method for ultimate load design of concrete structures. The steps followed in this method are:

1. An elastic analysis is done for all possible combinations of loading and envelope moment diagram is drawn.

2. These moments are adjusted to obtain a desirable ultimate moment distribution satisfying equilibrium and for economy or practicality of designing for such distributions.

3. The sections are designed and a bilinear moment curvature relation based on certain experimental or theoretical rule is obtained for each section.

4. For each possible critical combination of ultimate loading, using any set of adjusted moments which satisfies statics and falls within the M_u's provided, calculate the corresponding inelastic bending angles (k_p times inelastic moment area) for each plastic region.

5. Calculate residual moment set by imposing these angles to be concentrated at the critical sections.

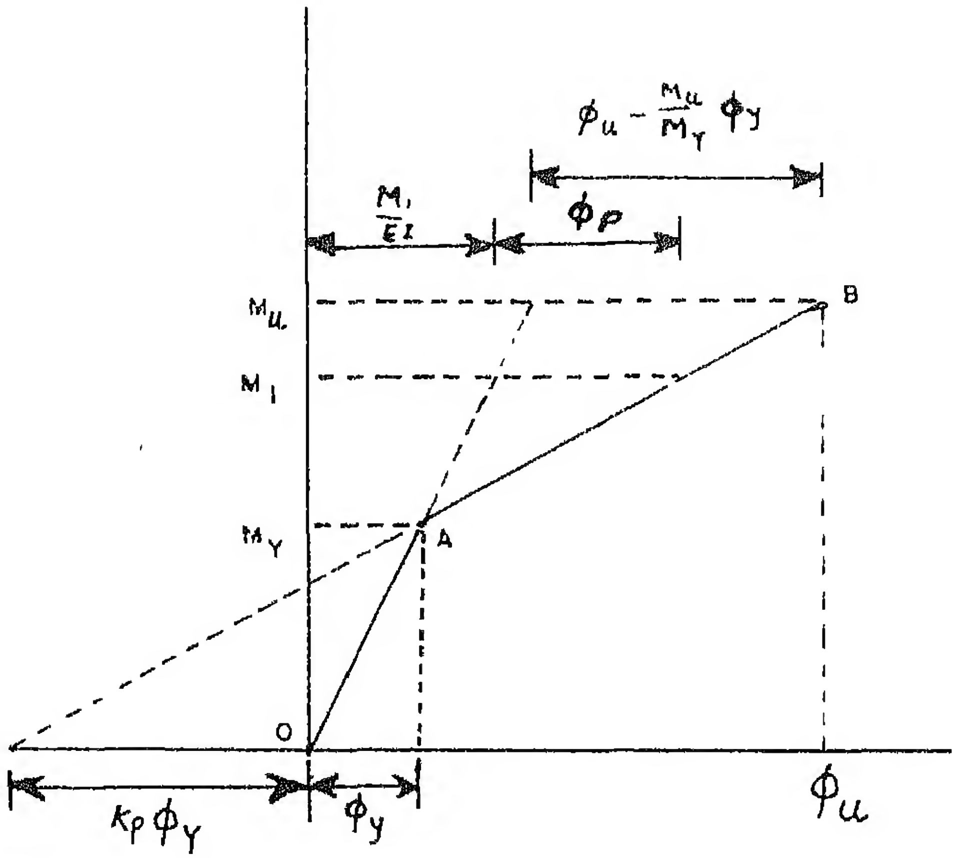
6. Superimpose the moments in step 5 on moments of step 2.

7. Check if the yield condition is satisfied at all the critical sections. If it is not satisfied, repeat steps 4 to 6.

8. Check the serviceability criterion using elastic analysis.

In brief, the basic justification of bilinear analysis and design[7,15,16,17,18] lies in taking account of all the reserve strength between elastic limit and ultimate stage accounting for strain hardening and yet not

exceeding the ductility limits. The two methods explained above are basically similar. In Macchi's method a trilinear M-θ model is used while in Sawyer's, a bilinear M-Ø relation is used. Nevertheless, both are based on the principle of ensuring ductility limits by imposing only allowable rotations as dislocations at the critical sections.



BILINEAR MOMENT CURVATURE RELATIONSHIP USED BY SAWYER

FIGURE 3.1

CHAPTER IV

INELASTIC ROTATION

4.1 Introduction

As discussed in Chapter III, in problems with limited ductility, it is essential to compute the inelastic rotations θ_i at the critical sections. This chapter is concerned with the evaluation of the inelastic rotation[19]. The operations are carried out in terms of the residual moments which remain at the critical sections when the structure is elastically unloaded from the state of collapse to the no-load condition.

4.2 Residual Moment System

Consider a structure which is loaded to a loading state $\lambda_n P$, when n hinges have formed in the structure, and it is about to collapse. If

M_{ai} = actual moment at section i at the loading stage $\lambda_n P$

M_{ei} = pseudo-elastic moment at section i due to load $\lambda_n P$ (assuming the structure to be behaving perfectly elastic during the entire loading history)

then the residual moment m_i is given by

$$m_i = M_{ai} - M_{ei} \tag{4.1}$$

Eq.(4.1) referred to hinge sections can be written as

$$m_i = M_{ui} - M_{ei} \tag{4.2}$$

If real hinges are introduced at all the locations of plastic hinges, rotation discontinuity occurs at these

hinges due to the moment system m_1. These rotations are the same as the actual inelastic rotations θ_1 occurring at the plastic hinges under the load $\lambda_n P$.

4.3 Energy of the Residual System

Consider a member jk between critical sections j and k, Fig.4.1. If

m_j = residual moment at end j and

m_k = residual moment at end k,

the strain energy in the member is given by

$$U_{jk} = \frac{L}{6EI} K_{jk}(m_j^2 + m_j\, m_k + m_k^2) \qquad (4.3)$$

where

U_{jk} = Energy in member jk

L = Unit dimension of length

K_{jk} = Relative stifness of the member jk

then the strain energy U_r for the whole structure is

$$U_r = \frac{L}{6EI} \sum K_{jk}(m_j^2 + m_j\, m_k + m_k^2)$$

where summation is over all the members of the structure.

4.4 Location of Last Hinge

In a structure which has developed n plastic hinges, Eq.(4.2) gives n known residual moment components. Since the residual moments are in equilibrium with zero applied load, an equilibrium condition can always be found relating these n residual moments. In general this equilibrium

equation can be written as

$$a_1 m_1 + a_2 m_2 + \ldots + a_i m_i + \ldots + a_j m_j + \ldots + a_k m_k + \ldots$$
$$+ a_n m_n = 0 \tag{4.4}$$

This means that there are only (n-1) independent residual moment components while the remaining residual moment components depend on these (n-1) moments. With this the expression for rotation θ_i is

$$\theta_i = \frac{\partial U_r}{\partial m_i} = \frac{L}{6EI} \quad K_{jk} \quad (2m_j + m_k)\frac{\partial m_j}{\partial m_i} + (m_j + 2m_k)\frac{\partial m_k}{\partial m_i}$$
$$\ldots\ldots \tag{4.5}$$

In the above expression, m_i represents one of the independent residual moments. Therefore for determining θ_i values from Eq.(4.5), it is essential to know which of the m_i components are independent and which are dependent. Since it is required that the inelastic rotation at the last hinge section should be zero, the residual moment of this section must be a dependent component. Thus, if the location of the last hinge is known, all the inelastic rotations at other hinge sections can be uniquely determined.

Consider a structure which has n hinges. Let k represent the section where the actual last hinge forms. Also assume that all the residual moments at the hinges act in the same direction as the correct inelastic rotations (in all the cases, the correct direction for the inelastic rotations are known as they should be in the direction

opposing the ultimate moment reaction at the section) so that the correct θ_i values given by $\partial U_r/\partial m_i$ are all positive. Suppose, on the basic of an arbitrarily assumed last hinge at section j, all other rotations are computed and denoted by θ_i'. If the location of the assumed last hinge is wrong, the θ_i' values based on the wrong last hinge will be of incorrect magnitude with some of them negative in sign. Also the rotation θ_k' at the correct last hinge k will not be equal to zero. Therefore, θ_i' values of rotations must be corrected to have the resulting rotations all positive and rotation at the last hinge k, zero. This requires a correction $-\theta_k'$ to be applied to section k so that $\theta_k' - \theta_k' = 0$. Now if $-\theta_k'$ be imposed at hinge k, the rotations induced at the other hinges can be derived from the principle of virtual work. From the equilibrium equation (4.4) the virtual work equation is

$$a_1 m_1 + a_2 m_2 + \dots + a_i m_i + a_j m_j + a_k m_k + \dots + a_n m_n = 0 \qquad (4.6)$$

As a result, the virtual rotations induced at i due to a rotation $-\theta_k'$ imposed at K is given by

$$-\theta_k' \frac{m_k}{m_i} = -\theta_k' \frac{a_i}{a_k}$$

In general the final correct rotations are

$$\theta_i = (\theta_i' - \theta_k' \frac{a_i}{a_k}) \qquad (4.7)$$

For this term to be positive or zero for all hinge section

$$\frac{\theta_i'}{a_i} - \frac{\theta_k'}{a_k} \geqslant 0 \tag{4.8}$$

for this condition to be satisfied θ_k'/a_k must be algebraically least (largest negative value). of all θ_i'/a_i values.

4.5 Summary of Steps to be Followed in Residual Moment Method

The basic steps involved in determining the inelastic rotations by the residual moment method are

1. Determine M_{ei} by multiplying the unit load elastic moment co-efficient by ultimate load $\lambda_u P$.

2. Find the residual moment m_i at all the hinge locations by subtracting M_{ei} from M_{ui}.

3. Correct direction is assigned to residual moments, so that they act in the expected direction of inelastic rotations.

As a general rule
adjusted sign = (-).(sign of M_{ui}).(sign of m_i before adjusting)

4. The equilibrium equations relating the residual moments of a mechanism is written. This also determines the virtual work co-efficient a_i.

5. Arbitrarily assume any section as last hinge. For example j and determine derivative $\partial m_j/\partial m_i$.

6. Obtain an expression for θ_i' using the Eq.(4.5).

7. If the degree of statical indeterminacy is n, and n+1 hinges from (i.e. if the collapse mechanism is complete) proceed to step 9. Otherwise choose as many independent critical sections as the number of remaining indeterminacy (say m_{11}, m_{12}...).

8. Equate inelastic rotation at these sections

$$\frac{\partial U_r}{\partial m_{11}} = 0, \quad \frac{\partial U_r}{\partial m_{12}} = 0 \quad \ldots$$

Solve for residual moments m_{11}, m_{12}, ... in terms of known residual moments.

9. Substitute all values of m_i in expressions obtained in step 7 and obtain θ_i' values.

10. If θ_i' values are all positive, then the correct rotation is $\theta_i = \theta_i'$. If not, find algebraically least value of θ_i'/a_i, thus determining the actual position of the last hinge 'k'.

11. Then the correct rotations are

$$\theta_i = \theta_i' - \theta_k \frac{a_i}{a_k} .$$

Example 4.1

The method of residual moment is applied to a simple fixed beam, Fig.4.2(a), to obtain the inelastic rotations at collapse.

The fixed beam is of constant flexural rigidity and and constant ultimate moment capacity (± 22.0).

The load factor λ_u at collapse is calculated by standard virtual work method

$$\lambda_u = \frac{(1 + 3/2 + \frac{1}{2}) \times 22}{20} = 3.3$$

Fig.4.2(b) represents the moment diagram at collapse and the elastic moment M_{ei} for load factor $\lambda_u = 3.3$ is shown in Fig.4.2(c). The residual moment calculation is given in Table 4.1 below

TABLE 4.1 Residual Moment Calculation for Example 4.1

Section	1	2	3
Plastic Moment M_{ui}	-22.0	22.0	-22.0
Elastic Moment M_{ei}	-29.3	19.57	-14.65
$M_{ui} - M_{ei}$	7.3	2.43	- 7.35
Residual moment with adjusted sign m_i	7.3	-2.43	- 7.35

<u>Evaluation of θ_i values:</u>

With reference to Fig.4.2(e), the equilibrium equation can be expressed in terms of m_i as

$$2m_1 + 3m_2 + m_3 = 0 \tag{4.9}$$

The strain energy U_r is given by

$$U_r = \frac{L}{6EI}(m_1^2 + m_3^2 + m_1 m_3)$$

Assuming section 1 to be location of last hinge, such that m_2 and m_3 are independent while m_1 is dependent; Eq.(4.9)

can be written as

$$m_1 = \frac{-3}{2} m_2 - \tfrac{1}{2} m_3$$

Therefore

$$\frac{m_1}{m_2} = -\frac{3}{2} \; ; \qquad \frac{m_1}{m_3} = -\tfrac{1}{2}$$

and $\theta_2' = \dfrac{U_r}{m_2} = \dfrac{L}{6EI}\left(-3m_1 - \tfrac{3}{2}m_3\right)$

and $\theta_3' = \dfrac{U_r}{m_3} = \dfrac{L}{6EI}\left(\tfrac{3}{2}\, m_3\right)$ (4.10)

Substituting for m_1 and m_3 from Table 4.1

$$\theta_1' = 0.0$$

$$\theta_2' = -\frac{10.97\ L}{6EI} \qquad (4.11)$$

$$\theta_3' = -\frac{11.0\ L}{6EI}$$

Since the assumption that last hinge forms at section 1, yields negative inelastic rotation at section 2 and 3; the assumed location of last hinge is incorrect.

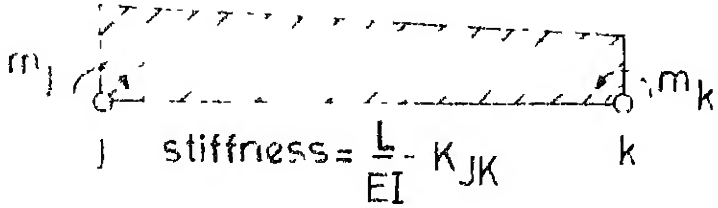
Now $\dfrac{\theta_2'}{a_2} = -\dfrac{10.97\ L}{18EI}$ and $\dfrac{\theta_3'}{a_3} = -\dfrac{11\ L}{6\ EI}$

hence the correct location of last hinge is section 3 where θ_i'/a_i is minimum. The correct inelastic rotations using Eq. (6.7) are

$$\theta_1 = 0 + \frac{\theta_3'}{a_3} \cdot a_1 = -\frac{3m_3\ L}{6EI} = \frac{22.05}{EI} \qquad (4.12)$$

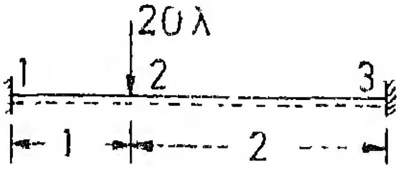
$$\theta_2 = \frac{L}{6EI}(-3m_1 - 6m_3) = \frac{22.20}{EI} \qquad (4.13)$$

$$\theta_3 = 0.$$

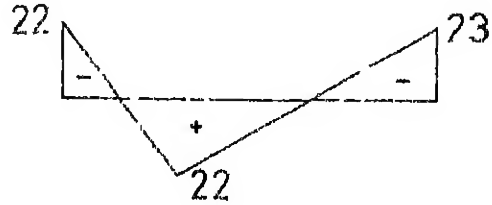


(a) , Typical Member with Residual Moment

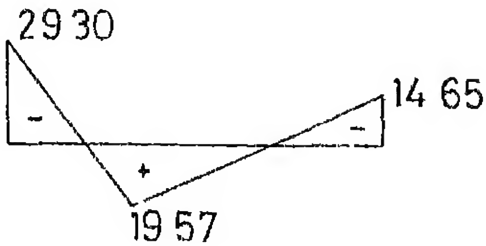
Figure 4 1



(a)

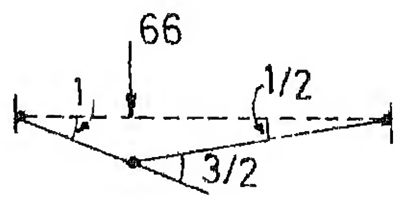


(b) - Mui

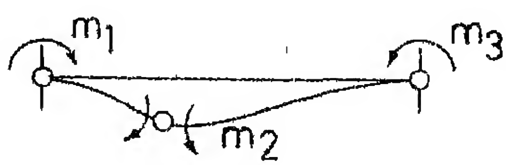


(c) - Mei

Elastic Moment Due to 20 λ_u



(d) Collapse Mechanism



(e) Direction of Inelastic Rotation

Figure 4 2 Fixed Beam for Example 4-1.

CHAPTER V

APPLICATION OF LINEAR PROGRAMMING TO PLASTIC ANALYSIS AND DESIGN

5.1 Introduction

The use of linear programming approach to plastic analysis was first pointed out by Charnes and Greenberg[24]. Later, Dorn and Greenberg[25] developed systematic procedure and illustrated it through a simple indeterminate truss example. Many other investigators[26,27,28,29,30,31] discussed the relationship between limit analysis and linear programming. The main advantage in plastic analysis of structures by linear programming is that highly indeterminate structures can be analysed easily by a digital computer.

5.2 General Problem of Linear Programming

A linear programming problem can be mathematically stated as

$$\text{Minimise or Maximise } Z = C_1X_1 + \ldots + C_nX_n \tag{5.1}$$

Subject to constraints

$$\sum_{j=1}^{n} a_{ij}\, X_j \; (\leqslant \text{or} \geqslant \text{or} =)\; b_i \;; \quad i = 1,\ldots m \tag{5.2}$$

and $$X_j \geqslant 0 \qquad j = 1,\ldots,n \tag{5.3}$$

where X_j, $j = 1,\ldots n$ are variables of the system and (5.2) and (5.3) represent constraints on interaction of

system parameters X_j. The non-negativity constraint expressed by Eq.(5.3) can be handled in an easy manner in simplex method and hence has not been included in (5.2).

5.3 Standard Form for Linear Programming Problem

By addition of certain variables (called slack and surplus variables) and some suitable transformation for those X which are unrestricted, the above linear programming problem is reduced to the form

$$\text{minimise } Z = \sum_{j=1}^{n} C_j X_j \qquad (5.4)$$

Subject to the constraints

$$\sum_{j=1}^{nv} a_{ij} X_j = b_i \qquad i = 1,2 \ldots m \qquad (5.5)$$

where nv is the total number of variables = No. of design variables + no. of slack variables + no. of surplus variables

and $X_j \geqslant 0 \qquad j = 1, \ldots, nv \qquad (5.6)$

5.4 The Simplex Method

The simplex method[21,22] is a two phase procedure for finding out an optimal solution of linear programming problems. A linear programming problem may have

(a) no feasible solution

(b) an unbounded solution

(c) finite number of optimal solutions

(d) infinite number of optimal solutions.

Phase I provides an initial basic feasible solution to the system of equations expressed by (5.5) and (5.6). If the constraint equations are inconsistent, the problem has no feasible solution. This condition is reflected at the end of Phase I.

Phase II uses the basic feasible solution obtained in Phase I as a starting point and finds either a optimum solution or yields the information that the design space is unbounded.

5.5 Application fo Plastic Analysis

A general plastic analysis problem can be written as

$$\text{Maximise} : Z = \lambda \tag{5.8}$$

Subject to the following constraints

(a) Equilibrium equation,

$$\sum_{j=1}^{n} a_{ij} M_j + a_{i,n+1} \lambda = 0 \qquad i = 1, \ldots e \tag{5.9}$$

where e = number of equilibrium equations

and n = number of critical sections.

(b) Yield Condition

$$M_{pj}^{-} \leq M_j \leq M_{pj}^{+} \qquad j = 1, \ldots .n \tag{5.10}$$

where M_j's are the moments at the critical sections ;

λ is load factor to be maximised;

$a_{i,j}$'s are the constants which are obtained from statics,

and M_{pj}^{+} and M_{pj}^{-} are the positive and negative plastic moment capacities respectively of critical section j.

Using the transformations

$$X_j = M_j - M_{pj}^{-} \quad \text{and} \quad X_{n+1} = \lambda \tag{5.11}$$

the Eq.(5.9) and Eq.(5.10) are transformed to

$$\sum_{j=1}^{n+1} a_{ij} X_j = B_i \tag{5.12}$$

$$i = 1, \ldots e$$

and

$$X_j \leqslant M_{pj}^{+} + M_{pj}^{-} \tag{5.13}$$

$$j = 1, \ldots n$$

where $B_i = \sum_{j=1}^{n} a_{ij} M_{pj}$

The plastic analysis problem in standard form is

$$\text{Minimise} : Z = - X_{n+1} \tag{5.14}$$

Subject to the constraints

$$\sum_{j=1}^{n+1} a_{ij} X_j = B_i \qquad i = 1, \ldots e \tag{5.15}$$

$$X_j + X_{n+j+1} = M_{pj}^{+} + M_{pj}^{-} \tag{5.16}$$

$$j = 1, \ldots n$$

$$X_j \geqslant 0 \qquad j = 1, \ldots 2n+1 \tag{5.17}$$

where X_{n+j+1} are slack variables added to inequalities in Eq.(5.13).

5.5.1 Nature of Solution

There are (n+e) constraints expressed by Eq.(5.15) and Eq.(5.16) and 2n+1 variables in the system. Thus atmost only

(n+e) of these 2n+1 variables will be non zero in any basic feasible solution. Again, since in physical problems, the load factor X_{n+1} cannot be zero, it will be a basic variable in the optimum solution. For other variables the following condition can occur:

(a) Some X_j's are in basis but the corresponding slack variables X_{n+j+1} are not in basis, then $X_{n+j+1} = 0$ and from Eq. (5.16),

$$X_j = M^+_{pj} - M^-_{pj}$$

Using Eq.(5.11), moment at section j is

$$M_j = M^+_{pi}$$

Hence, the moment at section j will be equal to the positive moment capacity of section.

(b) Some X_j are not in basis, then the corresponding slack variable X_{n+j+1} will be in basis i.e.

$$X_j = 0 \qquad \text{and} \qquad X_{n+j+1} = M^+_{pj} - M^-_{p}$$

again using Eq.(5.11),

$$M_j = - M^-_{pj}$$

Hence, in this case moment at section j equals negative moment capacity of section.

(c) Both X_j and X_{n+j+1} are basic variables and both are not zero. In such a case moment at critical section will be less than the plastic moment capacity of the section j.

(d) Both X_j and X_{n+j+1} are in basis but one is zero, i.e., the basic feasible solution is degenerate. In such a case the moment at section j will be equal to the plastic moment capacity.

Assuming that there are K number of sections for which either X_j or X_{n+j+1} is not in basis, the remaining 2(n-K)+1 variables will all be in basis. However, the total number of variables in the basis must be equal to the number of constraints and hence

$$2(n-K) + 1 + K = n + e$$

or $$K = n - e + 1 \tag{5.18}$$

Evidently (n-e) equals the number of indeterminacies of the structure considered. Hence, the following statement is made regarding the nature of a optimum basic feasible solution of a plastic analysis problem.

"Any optimum basic feasible solution will always have the moment equal to the moment capacity of critical sections atleast at (n-e+1) sections. In other words if m is a number of indeterminacy, simplex algorithm will always show formation of at least (m+1) hinges in optimum solution. However, all these hinges may not be involved in the collapse mechanism (as in the case of a partial collapse)".

As it is easily seen, if condition (d) exists, number of hinging section will be more than (m+1).

This property of optimum solution can be used to obtain a possible design solution in cases of partial collapse.

Example 5.1

Consider the single storey portal frame shown in Fig.5.1(a). The frame has five critical sections. All members are of uniform and equal plastic moment capacity.

The equilibrium equations are expressed as

$$-M_2 + 2M_3 - M_4 = 150\lambda$$
$$-M_1 + M_2 - M_4 + M_5 = 150\lambda$$

and the yield condition is

$$-100 \leq M_i \leq 100 \qquad i = 1,\ldots 5 \tag{5.18}$$

Substituting $M_i = X_i - 100$

and $\lambda = X_6$ (5.19)

the plastic analysis problem is expressed as

$$\text{Maximise} : Z = X_6$$

Subject to the constraints

$$\begin{aligned} X_2 - 2X_3 + X_4 + 150\, X_6 &= 0 \\ X_1 - X_2 \qquad + X_4 - X_5 + 150X_6 &= 0 \\ X_i \leq 200 \qquad i=1\ldots 5& \\ X_i \geq 0 \qquad i=1,\ldots 5& \end{aligned} \tag{5.20}$$

adding slack variables, the problem in standard form is expressed as

$$\text{Minimise} \quad Z = -X_6$$

Subject to

$$
\begin{array}{lllllllllll}
 & X_2-2X_3+X_4+150X_6 & & & & & & & = 0 \\
X_1-X_2 & +X_4-X_5+150X_6 & & & & & & & = 0 \\
X_1 & & +X_7 & & & & & & = 200 \\
 & X_2 & & +X_8 & & & & & = 200 \\
 & & X_3 & & +X_9 & & & & = 200 \\
 & & & X_4 & & +X_{10} & & & = 200 \\
 & & & & X_5 & & +X_{11} & & = 200
\end{array}
\qquad (5.21)
$$

where X_7, X_8, X_9, X_{10} and X_{11} are slack variables.

The solution of this linear programming problem obtained by simplex algorithm is

$$
\begin{array}{lll}
X_1 = 0 \ ; & X_2 = 100; & X_3 = 200 \\
X_4 = 0 \ ; & X_5 = 200 \ ; & X_6 = 2
\end{array}
\qquad (5.22)
$$

by back transformation using Eq.(5.19)

$$
\begin{array}{lll}
M_1 = -100 \ ; & M_2 = 0.0 \ ; & M_3 = 100.0 \\
M_4 = -100.0; & M_5 = 100.0 &
\end{array}
\qquad (5.23)
$$

Load factor $= 2.0$

This moment distribution is shown in Fig.5.1(b).

5.6 Application to Plastic Design Problems

A plastic design problem can be stated to be as

"Given the ultimate load, the structure has to withstand, to find out the plastic moment capacity of various members in such a way that cost or weight of structure is minimum".

The formulation of plastic design problem is illustrated by the following example.

Example 5.2

The frame shown in Fig.5.2(a) consists of three members. The weight of members are assumed to be proportional to its plastic moment capacity. Further, to simplify the procedure the moment capacity of two sections are assumed to be equal. With these assumptions the objective function for design problem can be written as

$$\text{Minimise} \quad Z = 4M_{p1} + 6M_{p2} \tag{5.24}$$

<u>Constraints on M_{p1} and M_{p2}:</u>

The possible critical mechanisms for problem are shown in Fig.5.2(b),(c),(d),(e),(f) and (g). The limit equilibrium condition for these mechanisms are

$$\begin{aligned}
&\text{for mechanism (b)} \quad && 4M_{p2} \geqslant 3 \\
&\text{for mechanism (c)} \quad && 4M_{p1} \geqslant 6 \\
&\text{for mechanism (d)} \quad && 2M_{p1}+4M_{p2} \geqslant 9 \\
&\text{for mechanism (e)} \quad && 2M_{p1}+2M_{p2} \geqslant 6 \\
&\text{for mechanism (f)} \quad && 2M_{p1}+2M_{p2} \geqslant 3 \\
&\text{for mechanism (g)} \quad && 4M_{p1}+2M_{p2} \geqslant 9
\end{aligned} \qquad \ldots \quad (5.25)$$

Out of these constraints if the first four relations are satisfied, the rest two will obviously be satisfied. Hence, finally the minimum weight design can be expressed in linear programming form as

$$\text{Minimise } Z = 4M_1 + 6M_2$$

Subject to the constraints

$$\begin{aligned} 4M_p \quad & \geq 6 \\ 4M_{p2} \quad & \geq 3 \\ 2M_{p1} + 4M_{p2} \quad & \geq 9 \\ 2M_{p1} + 2M_{p2} \quad & \geq 6 \end{aligned} \tag{5.26}$$

The solution of this problem using simplex algorithm is

$$M_{p1} = \frac{3}{2} \; ; \quad M_{p2} = \frac{3}{2} \; ; \quad Z = 15 \tag{5.27}$$

5.7 Problem of Partial Collapse

For more complex design problems, it is very difficult to conceive of all the possible mechanisms of a plane frame. Due to this fact the designer is forced to select some preliminary moment capacity of members and adjust them to obtain a desired load factor. However, if the preliminary plastic moment capacities lead to a partial collapse of the frame, then it is a very difficult task to obtain a bending moment distribution satisfying yield condition by classical methods of plastic analysis. This difficulty can be easily overcome by using linear programming formulation of analysis problem. Thus linear programming can be very conveniently used to obtain a

better possible design. The steps suggested in this direction are.

1. Obtain moment capacities of beams based on beam mechanisms and of columns based on panel mechanisms.

2. Using preliminary moment capacities obtained in step 1 find out the collapse load factor 'λ' and the corresponding moment distribution M_1. If the 'λ_a' is actual load factor required for the frame, then one possible design solution for frame is

$$M_{pj} = \frac{\lambda_a}{\lambda} \times \text{Maximum moment in Member } j$$

where M_{pj} is plastic moment capacity of member j.

It can be easily seen that this will lead to a better design than obtained in step 1.

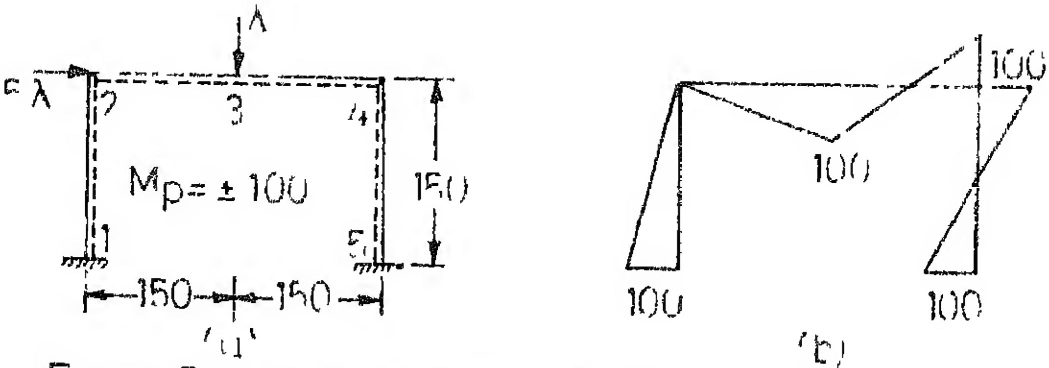
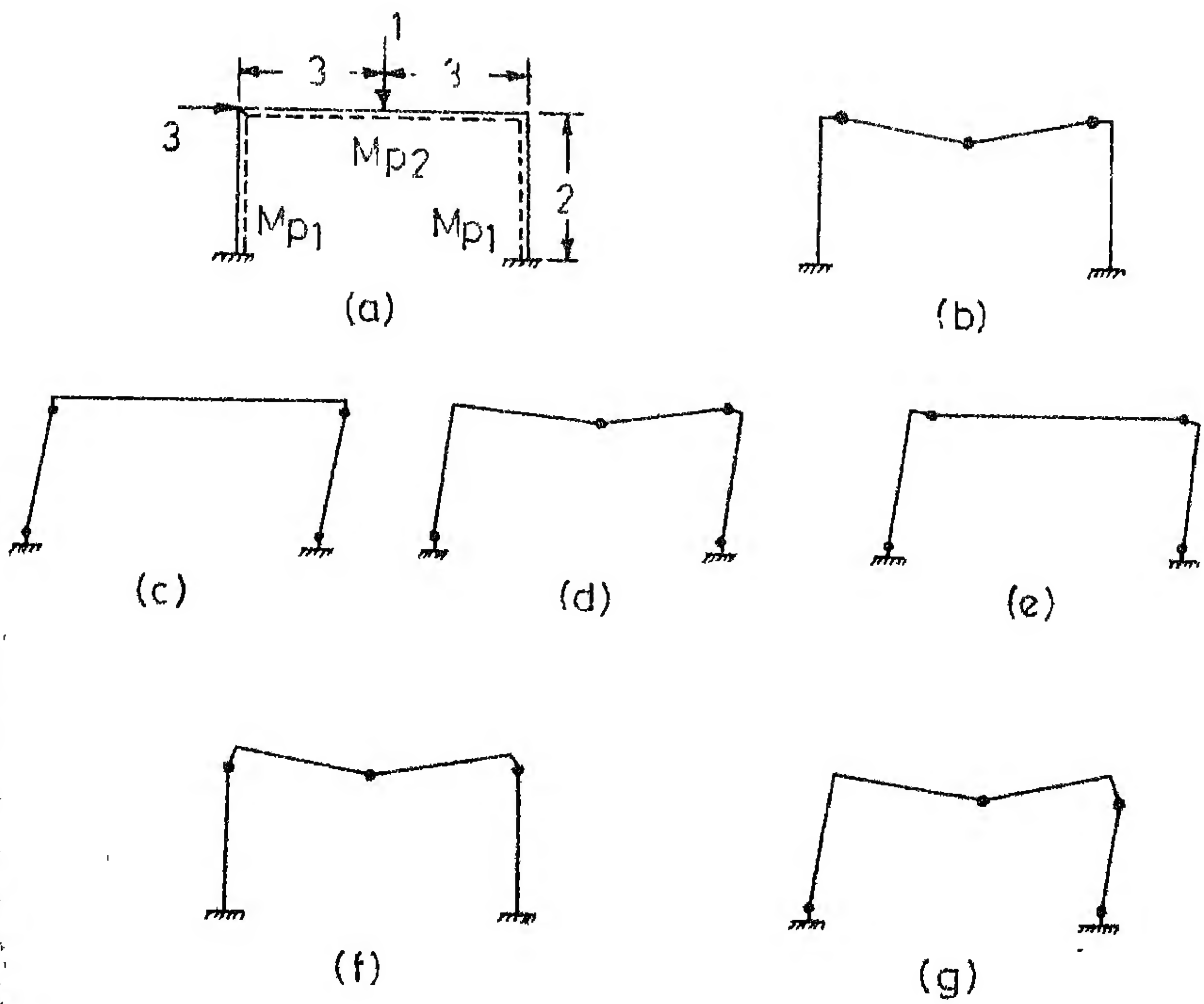


Figure 5.1 Portal Frame for Example 5.1



Frame for Example 5.2

Figure 5.2

CHAPTER VI

PROPOSED LIMIT THEORY ACCOUNTING STRAIN HARDENING AND DUCTILITY EFFECT

6.1 Introduction

As mentioned in earlier chapters, limit analysis based on bilinear M-ϕ relation with strain hardening effect is quite complex. In this chapter, a method of limit analysis using linear programming technique and accounting for strain hardening effects with limited ductility is proposed. It is an automatic procedure where in the three conditions, namely

(a) equilibrium (b) yield condition and (c) rotation compatibility are simultaneously satisfied. The method is ideally suited for computers.

6.2 Moment-Curvature Relationship

In order to define a bilinear M-ϕ relationship (Fig.6.1(a)), four quantities, M_u, EI, α and B are required. For the typical bilinear M-ϕ curve in Fig.6.1(a) line OA, through origin, represents the elastic line for which the curvature is given by

$$\phi = \frac{M}{EI} \qquad (6.1)$$

where M is the moment at the point under consideration. For any moment M_1, between M_u and M_y, the total curvature

can be considered to consist of two components

(a) pseudo-elastic component, M_1/EI

(b) excess in curvature, ϕ_p, caused by inelasticity.

From Fig.6.1(a), the value of inelastic curvature ϕ_p can be expressed as

$$\phi_p = \left(\frac{B - \alpha}{\alpha - 1}\right)\left(\frac{M_1 - M_y}{EI}\right) \qquad (6.2)$$

where B is called the ductility factor and is given by

$$B = \frac{\phi_u}{\phi_y} \qquad (6.3)$$

and α is the ratio of moments M_u and M_y corresponding to point A, Fig.6.1(a).

6.3 Rotation Capacity 'θ_p' of Critical Sections

By definition, rotation capacity θ_p is a measure of the ability of a section to withstand an applied inelastic rotation. Hence a limitation on θ_{pi} also implies a limitation on ductility. As the loads on the structure are gradually increased after a critical section reaches its M_y value, inelasticity gradually spreads around the critical section. At any instant the length of this inelastic region is referred to as "inelastic length, l", Fig.6.2.

Fig.6.3(a) and (b) show a typical moment distribution around a critical section and the corresponding curvature distribution. The inelastic region is indicated by the shaded region. The rotation capacity θ_p of a section can be expressed as

$$\theta_p = \int_0^l \phi_p \, dx \tag{6.4}$$

where ϕ_p represents the inelastic curvature distribution within the length l. The quantity θ_p corresponds to the shaded area in Fig.6.3(b). Using Eq.(6.3), the expression for θ_p can be written as

$$\theta_p = \int_0^l \left(\frac{B - \alpha}{\alpha - 1}\right) \left(\frac{M - M_y}{EI}\right) dx$$

assuming the section properties, , B, EI and M_y to be the same throughout the inelastic length l,

$$\theta_p = \left(\frac{B - \alpha}{\alpha - 1}\right) \frac{1}{EI} \int_0^l (M - M_y) \, dx \tag{6.5}$$

The dominant parameters influencing rotation capacity are

(a) hinge length, l.

(b) the ratio of ultimate curvature ϕ_u and yield curvature ϕ_y, B.

(c) elastic stiffness, EI

(d) shape of the moment diagram around the critical sections.

(e) the ratio of moments M_u and M_y, α .

While α, B, EI and M_u depend on the properties of the sections, hinge length l and moment distribution depends on the loading pattern and behaviour of the structure as a whole. However, the parameter EI cancels from both sides

of the compatibility condition $\theta_i \leq \theta_{pi}$ and hence its absolute value has no influence on the problem of compatibility.

6.4 Limit Analysis of Plane Frames

Based on the assumptions stated in Chapter I and the bilinear M-ϕ curve (Fig.6.1) for critical sections, the problem of limit analysis is stated as

" Given the bilinear M-ϕ property of the critical sections, to find the maximum load which the structure can withstand without overstraining the material".

Mathematically, this problem can be stated as

Maximise · Z = Load factor λ

Subject to the constraints

(a) Equilibrium equations expressed as Eq.(5.9)

(b) Yield criterion (Eq.5.10)

(c) Rotation compatibility specifying that

$$\theta_i \leq \theta_{pi} \qquad (6.9)$$

$$i = i_1, i_2, \ldots i_p$$

where $i_1, i_2, \ldots i_p$ denote the sections where inelasticity has been attained.

For an analysis problem, the configuration of structure, loading pattern, and section properties such as M_u, EI, α and B are assumed known for all critical sections. Hence, all the constraints discussed earlier

can be expressed explicitly in terms of moments at critical sections; load factor λ and other parameters. The complexity of the problem lies in the formulation of constraints in Eq.(6.9) where explicit expressions for θ_1 and θ_{p1} cannot be obtained unless the load factor and the corresponding moment distributions are known. This necessitates the adoption of some iterative procedure for analysis.

The proposed method consists of finding the ultimate load and corresponding moment distribution, first without considering the rotation compatibility constraint. Next the method of residual moment[19] is used to obtain an expression for inelastic rotation at critical sections. The proper sign convention, as explained in Chapter IV, assures that all the inelastic rotations will always be positive. Having obtained the expression for inelastic rotation, the rotation capacities at critical sections are calculated. The rotation compatibility constraint is now introduced in the linear programming problem already formulated. In the second cycle of operation, the ultimate load is found for this new mathematical model of ultimate load analysis problem. If more accuracy is desired, a third cycle can be carried out with slight modification in θ_{p1} values of the rotation compatibility constraint only.

6.5 Summary of Steps in the Proposed Analysis Procedure:

The basic steps involved in determining the ultimate load for inelastic structures are:

1. Form the equilibrium equation and the constraints on moments

2. By suitable transformation (Eq.5.11) and addition of slack and surplus variables, express the constraints in the standard form (Eq.5.7).

3. Maximise, using linear programming, the load factor λ_u' subject to constraints formulated in step 2.

4. Obtain, by any elastic analysis procedure, an elastic moment distribution for an unit load on the structure.

5. Calculate pseudo-elastic moment M_{ei} at load $\lambda_u' P$, where P is unit load on structure (assume the structure to behave in a perfectly elastic manner during entire history of loading).

6. Using the method of residual moment distribution, obtain expressions for inelastic rotation at those critical sections where ultimate moment has developed during step 3. Thus

$$\theta_i = \sum_{j=1}^{p} a'_{i,j} M_j + a'_{i,n+1} \lambda \qquad (6.10)$$

where j's are only those sections where inelastic rotation is not zero and a_{ij}'s are constants. These expressions for inelastic rotations are p in number,

where p = Number of indeterminacy.

7. Using the Eq.(6.3) and Eq.(6.4) and moment distribution obtained in step 3, obtain the rotation capacity of the critical section.

8. Append the constraint $\theta_1 \leq \theta_{p1}$ to the already formulated constraints.

With the sign convention used in step 6, both θ_1 and θ_{p1} should always be positive for consistency in deformation. However, if rotation compatibility constraint is $\theta_1 \leq \theta_{p1}$ and load factor is maximised the final moment distribution can give a negative value of θ_1. This inconsistency in deformation is removed by introduction of additional constraints expressed as

$$\theta_1 \geq 0 \qquad (6.11)$$

Thus the mathematical model for limit analysis problem is

Maximise Z = load factor λ

Subject to

(a) Equilibrium equation

(b) Yield criterion

(c) Rotation compatibility, Eq.6.10

(d) Constraint ensuring consistency in rotation, Eq. (6.11).

9. Obtain the maximum load factor and the corresponding moment distribution using simplex algorithm.

10. If more accuracy is required, the θ_{pi} values in constraint Eq.(6.9) are changed corresponding to the moment distribution obtained in step 9, and the inequality sign in constraint (6.9) for these i where the moment has become less than M_{yi}, is changed to equality. The corresponding expression in Eq.(6.11) is removed and simplex algorithm is again used for finding the ultimate load with respect to newly formed constraints.

Examples

The proposed method of ultimate load analysis is illustrated by three problems

(a) a fixed beam

(b) single storey, single bay portal frame

(c) single storey, two bay portal frame.

The moment curvature relations for the critical sections are assumed to be same in positive and negative moment. However, the method is general and can be applied to variation in this property.

Example 6.1:

It is required to find out the ultimate load and corresponding moment distribution for a fixed end beam shown in Fig.4.2(a). The bilinear M-Ø property of critical sections are defined as

$M_u = 22$

$B = 6.1$

$\alpha = 1.1$

$EI = 2900.$

Inelastic Curvature:

For any moment M, between M_y and M_u, inelastic curvature is

$$\phi_p = \left(\frac{B - \alpha}{\alpha - 1}\right)\left(\frac{M - M_y}{EI}\right)$$

$$= \frac{50(M - 20)}{EI} \qquad (6.12)$$

First Cycle of Analysis:

The problem of ultimate load analysis is

$$\text{Maximise} : Z = \lambda \qquad (6.13)$$

Subject to

(a) Equilibrium equation

$$2M_1 - 3M_2 + M_3 + 40\lambda = 0 \qquad (6.14)$$

(b) Yield Criterion

$$-22 \leq M_i \leq 22 \qquad (6.15)$$

$$i = 1, \ldots 3$$

The optimum solution of this problem is

$$M_1 = -22, \quad M_2 = 22, \quad M_3 = -22, \quad \lambda = 3.3 \qquad (6.16)$$

This moment distribution is shown in Fig.4.2(b).

From Table 4.1, it is seen that

$$m_1 = M_1 + 8.88\lambda$$

$$m_2 = -M_2 + 5.92\lambda$$

$$m_3 = M_3 + 4.44\lambda$$

Substituting these values in Eq.(4.12) and Eq.(4.13) for inelastic rotations

$$\theta_1 = \frac{-3M_3 - 13.33\lambda}{2EI}$$
$$\theta_2 = \frac{-M_1 - 6M_3 - 53.28\lambda}{2EI} \qquad (6.17)$$

Rotation Capacity:

$$\theta_{p1} = \tfrac{1}{2} \times 0.046 \times \frac{50}{EI} \times 2 = \frac{2.30}{EI}$$
$$\theta_{p2} = \tfrac{1}{2} \times 0.138 \times \frac{50}{EI} \times 2 = \frac{6.90}{EI} \qquad (6.18)$$
$$\theta_{p3} = \tfrac{1}{2} \times 0.092 \times \frac{50}{EI} \times 2 = \frac{4.60}{EI}$$

Rotation Compatibility Constraint:

From Eq.(6.17) and Eq.(6.18), rotation compatibility constraint are

$$-3M_3 - 13.33\lambda \leqslant 4.60$$
$$-3M_1 - 6M_3 - 53.28\lambda \leqslant 13.8 \qquad (6.19)$$

and for θ to be positive

$$-3M_3 - 13.33\lambda \geqslant 0$$
$$-3M_1 - 6M_3 - 53.28\lambda \geqslant 0 \qquad (6.20)$$

Second Cycle of Analysis:

The maximum value of λ, subject to constraints expressed by Eq.(6.14), (6.15), (6.19) and (6.20) is tabulated in Table 6.1.

TABLE 6.1

Section	1	2	3
Moment, M	-22	18.10	13.95
Inelastic Rotation θ_1	4.58	0.0	0.0
Rotation Capacity θ_{p1}	5.04	0.0	0.0
Ultimate Load factor λ = 2.8			

The above solution has been obtained using a standard, 'share' subroutine for minimization of a function by simplex algorithm. It is seen that the actual load factor is 15% less than that predicted by mechanism method of analysis.

Example 6.2

It is required to find out the ultimate load for the frame in Fig.6.4(a). The Section properties are

$M_u = 22$

$B = 3.6$

$\alpha = 1.1$

$EI = 2900$

Inelastic Curvature

For any moment M between M_y and M_u, the inelastic curvature is

$$\phi_p = \frac{3.6 - 1.1}{0.1} \times \frac{(M - 20)}{EI} = \frac{25}{EI}(M-20) \tag{6.21}$$

First Cycle of Analysis

The ultimate load analysis problem without compatibility constraint is

Maximize : $Z = \lambda$

Subject to the constraints

$$M_2 - 2M_3 + M_4 + 20\lambda = 0$$

$$M_1 - M_2 + M_4 - M_5 + 30\lambda = 0$$

$$-22 \leqslant M_i \leqslant 22$$

$$i = 1, \ldots 5 \tag{6.22}$$

The solution of this problem is

$$M_1 = -22.0; \quad M_2 = 13.2; \quad M_3 = 22.0;$$
$$M_4 = -22.0; \quad M_5 = 22.0; \quad \lambda = 2.64 \tag{6.23}$$

Inelastic Rotations

The distribution of M_{ui}, M_{ei} and m_i are tabulated below in Table 6.2.

TABLE 6.2 Residual Moment Calculation

Section	1	3	4	5
M_{ui}	-22.0	22.0	-22.0	22.0
M_{ei}	-18.2	17.63	-25.7	26.95
$M_{ui}-M_{ei}$	- 3.8	4.37	3.7	-4.95
Residual moment m_i with adjusted sign	-3.8	-4.37	3.7	4.95

The positive direction of residual moments m_1 are shown in Fig. 6.4(d). The equilibrium equations in terms of residual moments are

$$m_1 + m_2 + \qquad m_4 + m_5 = 0$$

$$m_1 \qquad +2m_3 + 2m_4 + m_5 = 0 \tag{6.24}$$

From Eq.(6.24)

$$m_2 = -4.85 \tag{6.25}$$

Assuming the last hinge to form at Section 3, the inelastic rotations at Sections 2 and 3 should be zero and hence m_2 and m_3 are the dependent components of residual moment system.

Thus using Eq.(6.24),

$$\frac{\partial m_3}{\partial m_1} = -\tfrac{1}{2}\ , \quad \frac{\partial m_3}{\partial m_4} = -1\ , \quad \frac{\partial m_3}{\partial m_5} = -\tfrac{1}{2}$$

$$\frac{\partial m_2}{\partial m_1} = -1\ , \quad \frac{\partial m_2}{\partial m_4} = -1\ , \quad \frac{\partial m_2}{\partial m_5} = -1 \tag{6.26}$$

Rotation calculation for this problem is given in Table 6.3.

Expressions for residual moments in terms of actual moment at sections 1, 3,4 and 5, where hinges have formed are

$$m_1 = M_1 + 6.9\ \lambda$$

$$m_2 = -(M_1 + M_4 - M_5 + 26.85\lambda)$$

$$m_4 = M_4 + 9.75\ \lambda$$

$$m_5 = -(M_5 - 10.2\ \lambda) \tag{6.27}$$

TABLE 6.3 Rotation Calculation of Example 6.2

Member	U_{ij}	$\theta_1' = \frac{\partial U_r}{\partial m_1}$	$\theta_4' = \frac{\partial U_r}{\partial m_4}$	$\theta_5' = \frac{\partial U_r}{\partial m_5}$
1 - 2	$m_1^2+m_1m_2+m_2^2$	$m_1-m_2 = 1.15$	$-m_1-2m_2=13.50$	$-m_1-2m_2=13.50$
2 - 4	$m_2^2+m_2m_4+m_4$	$-2m_2-m_4=6.0$	$-m_2+m_4 = 8.55$	$-2m_2-m_4= 6.0$
4 - 5	$m_4^2+m_4m_5+m_5^2$		$2m_4+m_5 = 11.35$	$m_4+2m_5=13.60$
Total θ_i		7.15	34.4	31.10

Using these expressions for residual moments, inelastic rotations are:

$$\theta_1 = 4M_1 + 2M_4 - 3M_5 + 77.70\,\lambda$$

$$\theta_4 = 2M_1 + 6M_4 - 4M_5 + 113.1\,\lambda$$

$$\theta_5 = 3M_1 + 4M_4 - 6M_5 + 120.9\,\lambda \qquad (6.28)$$

The θ values are all relative values of inelastic rotations (actual value multiplied by 6EI/L).

Rotation Capacity

The inelastic lengths are

$$l_1 = 0.227, \quad l_3 = 0.54, \quad l_4 = 0.273, \quad l_5 = 0.181$$

Then relative values of rotation capacities (actual value multiplied by 6EI/l) are

$$\theta_{p1} = \tfrac{1}{2} \times 0.227 \times 50 \times \frac{6}{4} = 8.5$$

$$\theta_{p4} = \tfrac{1}{2} \times 0.273 \times 50 \times \frac{6}{4} = 10.2$$

$$\theta_{p5} = \tfrac{1}{2} \times 0.181 \times 50 \times \frac{6}{4} = 6.9 \qquad (6.29)$$

Second Cycle of Analysis

The additional constraints in second cycle of analysis are

$$4M_1 + 2M_4 - 3M_5 + 77.70\lambda \leqslant 8.5$$

$$2M_1 + 6M_4 - 4M_5 + 113.1\lambda \leqslant 10.2$$

$$3M_1 + 4M_4 - 6M_5 + 120.9\lambda \leqslant 6.9$$

$$4M_1 + 2M_4 - 3M_5 + 77.0\lambda \geqslant 0.0$$

$$2M_1 + 6M_4 - 4M_5 + 113.1\lambda \geqslant 0.0$$

$$3M_1 + 2M_4 - 3M_5 + 77.70\lambda \geqslant 0.0 \qquad (6.30)$$

Maximum value of λ and corresponding moment distribution, subject to constraint Eq.(6.22) and Eq.(6.30) is given in Table 6.4.

TABLE 6.4 Moment Distribution at Ultimate Load for Example 6.2.

Section	1	2	3	4	5
M	-17.67	8.1	16.3	-22.0	22.0
θ_1	0	0	0	7.66	6.9
θ_{p1}	0	0	0	10.7	6.9
Load factor = 2.32					

Reduction in limit load is 12.2 percent.

Example 6.3

It is required to find out the ultimate load for the frame in Fig.6.5(a). The section properties for all the critical sections are

$\alpha = 1.18$, $B = 6.0$, $EI = 2900$, $M_u = 10.0$

First Cycle of Analysis

With no restriction on inelastic rotation at critical sections, the ultimate load analysis problem is

Maximise $Z = \lambda$

Subject to the constraints

$$M_2 - 2M_3 + M_4 \qquad +20\lambda = 0$$

$$M_6 - 2M_7 + M_8 \qquad +20\lambda = 0$$

$$-M_4 + M_5 + M_6 \qquad = 0$$

$$M_1 - M_2 \qquad + M_5 \qquad + M_8 - M_9 - M_{10} + 20\lambda = 0$$

and $\quad -10 \leqslant M_i \leqslant 10$

$$i = 1, \ldots 10 \qquad (6.31)$$

The solution of this problem is

$M_1 = -10.0$; $\quad M_2 = -6.67$; $\quad M_3 = 10.0$; $\quad M_4 = -10.0$

$M_5 = -3.33$; $\quad M_6 = -6.67$; $\quad M_7 = 10.0$; $\quad M_8 = -10.0$

$M_9 = 10.0$; $\quad M_{10} = 10.0$; $\quad \lambda = 1.833 \qquad (6.32)$

Inelastic Rotations

The distribution of M_{ui}, M_{ei} and m_i are tabulated below in Table 6.5.

TABLE 6.5 Residual Moment Calculations for Example 6.3

Section	1	3	4	7	8	9	10
M_{ui}	-10.0	10.0	-10.0	10.0	-10.0	10.0	10.0
M_{ei}	- 4.2	10.9	-14.88	9.75	- 9.16	8.78	7.64
$M_{ui}-M_{ei}$	- 5.8	-0.9	4.88	0.25	- 0.84	1.22	2.36
Residual moment m_i with adjusted sign	- 5.8	0.9	4.88	-0.25	- 0.84	-1.22	-2.36

The positive direction of the residual moments m_i are shown in Fig.6.5(d). The equilibrium equations in terms of the residual moment distribution m_i are

$$
\begin{aligned}
m_1 + 2m_3 + 2m_4 + 2m_5 \qquad\qquad +2m_8+m_9+m_{10} &= 0 \\
m_2+2m_3 + m_4 \qquad\qquad\qquad\qquad &= 0 \\
m_6 + 2m_7 - m_8 &= 0 \\
m_4 - m_5 \qquad - m_8 &= 0
\end{aligned}
\tag{6.33}
$$

using these relations, residual moments m_2, m_5 and m_6 are

$$
\begin{aligned}
m_2 &= -6.68 \\
m_5 &= 3.54 \\
m_6 &= 1.34
\end{aligned}
\tag{6.34}
$$

Assuming the last hinge to form at Section 1, the inelastic rotations at Sections 1,2,5 and 6 should be zero and hence, m_1, m_2, m_5 and m_6 are dependent components of residual moment system. Thus using Eq.(6.33)

$$\frac{\partial m_1}{\partial m_3} = -2 \ ; \quad \frac{\partial m_1}{\partial m_4} = -2 \ ; \quad \frac{\partial m_1}{\partial m_7} = -2 \ ; \quad \frac{\partial m_1}{\partial m_8} = -2 \ ;$$

$$\frac{\partial m_1}{\partial m_9} = -1, \quad \frac{\partial m_1}{\partial m_{10}} = -1$$

$$\frac{\partial m_2}{\partial m_3} = -2 \ ; \quad \frac{\partial m_2}{\partial m_4} = -1 \ ; \quad \frac{\partial m_6}{\partial m_7} = -2, \quad \frac{\partial m_6}{\partial m_8} = -1$$

and $$\frac{\partial m_5}{\partial m_4} = -1 \ ; \quad \frac{\partial m_5}{\partial m_7} = 2 \ ; \quad \frac{\partial m_5}{\partial m_8} = 1 \tag{6.35}$$

Rotation calculation for this problem is given in Table 6.6.

TABLE 6.6 Rotation Calculation of Example 6.3

Member ij	U_{ij}	$\theta'_3 = \frac{\partial U_{ij}}{\partial m_3}$	$\theta'_4 = \frac{\partial U_{ij}}{\partial m_4}$	$\theta'_7 = \frac{\partial U_{ij}}{\partial m_7}$	$\theta'_8 = \frac{\partial U_{ij}}{\partial m_8}$	$\theta'_9 = \frac{\partial U_{ij}}{\partial m_9}$	$\theta_{10} = \frac{\partial U_{ij}}{\partial m_{10}}$
1,2	$m_1^2+m_1m_2+m_2^2$	$-6m_1-6m_2$ $= 75.88$	$-5m_1-4m_2$ $= 55.72$	$-4m_1-2m_2$ $= 36.56$	$-4m_1-2m_2$ $= 36.56$	$-2m_1-m_2$ $= 18.28$	$-2m_1-m_2$ $= 18.28$
2,4	$m_2^2+m_2m_4+m_4^2$	$-4m_2-2m_4$ $= 16.06$	$-m_2+m_4$ $= 10.56$	-	-	-	-
5, 10	$m_5^2+m_5m_{10}+m_{10}^2$	-	$2m_5+m_{10}$ $= 4.72$	$4m_5+2m_{10}$ $= 9.44$	$2m_5+m_{10}$ $= 4.72$	-	m_5+2m_{10} $=-1.18$
6, 8	$m_6^2+m_6m_8+m_8^2$	-	-	$-4m_6-2m_8$ $= -3.68$	$-m_6+m_8$ $= -2.18$	-	-
8, 9	$m_8^2+m_8m_9+m_9^2$	-	-	-	$2m_8+m_9$ $= -2.9$	m_8+2m_9 $= -3.24$	-
Total Rotation θ'_i $\theta_i = \theta'_i$		91.94	71.00	42.32	36.20	15.04	17.10

Expressions for residual moments in terms of actual moment at sections 1,3,4,7,8,9 and 10 where hinges have formed are

$$
\begin{aligned}
m_1 &= M_1 + 2.293\lambda \\
m_2 &= -2m_3 - m_4 = 2M_3 - M_4 + 20.016\lambda \\
m_3 &= -M_3 + 5.95\lambda \\
m_4 &= M_4 + 8.116\lambda \\
m_5 &= m_4 - m_6 = M_4 - 2M_7 + M_8 + 23.736\lambda \\
m_6 &= -2m_7 - m_8 = 2M_7 - M_8 - 15.626\lambda \\
m_7 &= -M_7 + 5.313\lambda \\
m_8 &= M_8 + 5.0\lambda \\
m_9 &= -M_9 + 4.785\lambda \\
m_{10} &= -M_{10} + 4.17\lambda
\end{aligned}
\qquad (6.36)
$$

Using these expressions for residual moments inelastic rotations are

$$
\begin{aligned}
\theta_3 &= -6M_1 - 20M_3 + 8M_4 + 170.17\lambda \\
\theta_4 &= -5M_1 - 10M_3 + 8M_4 - 4M_7 + 2M_8 - M_{10} + 159.1\lambda \\
\theta_7 &= -4M_1 - 4M_3 + 6M_4 - 16M_7 + 6M_8 - 2M_{10} + 185.6\lambda \\
\theta_8 &= -4M_1 - 4M_3 + 4M_4 - 6M_7 + 6M_8 - M_9 - M_{10} + 117.9\lambda \\
\theta_9 &= -2M_1 - 2M_3 + M_4 + M_8 - 2M_9 + 30.1\lambda \\
\theta_{10} &= -2M_1 - 2M_3 + 2M_4 - 2M_7 + M_8 - 2M_{10} + 47.5\lambda
\end{aligned}
\qquad (6.37)
$$

The θ values are all relative values of inelastic rotation (the absolute value = θL/6EI).

Rotation Capacity

Inelastic curvature for moment M between M_y and M_u is

$$\theta_p = \left(\frac{\beta - \alpha}{\alpha - 1}\right) \frac{(M - M_y)}{EI} = \frac{26.8}{EI} (M - 8.5) \qquad (6.38)$$

From Fig.6.5(c), the inelastic lengths are

$$l_3 = 0.329 \;; \quad l_4 = 0.15 \;; \quad l_7 = 0.329$$

$$l_8 = 0.45 \;; \quad l_9 = 0.30 \;; \quad l_{10} = 0.45 \qquad (6.39)$$

Then relative values of rotation capacities are (absolute value multiplied by 6EI/L)

$$\theta_{p3} = \frac{1}{2} \times 0.329 \times 26.8 \times 1.5 \times \frac{6}{4}$$

$$= 10.2$$

$$\theta_{p4} = 4.65$$

$$\theta_{p7} = 10.2$$

$$\theta_{p8} = 13.95$$

$$\theta_{p9} = 9.30$$

$$\theta_{p10} = 13.95 \qquad (6.40)$$

Second Cycle of Analysis

The limit analysis problem with rotation compatibility constraint is

Maximise : $Z = \lambda$

Subject to the constraints

(a) Equilibrium equations and yield condition, Eq.(6.21)

(b) $\theta_i \leqslant \theta_{pi}$

and $\theta_i \geqslant 0$

where θ_i and θ_{pi} are given by Eq.(6.37) and Eq.(6.40) respectively.

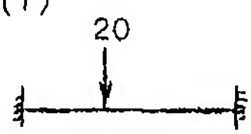
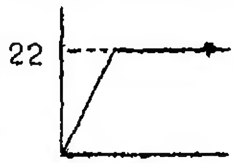
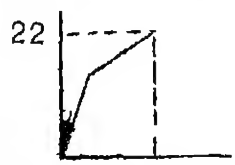
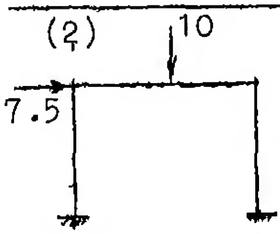
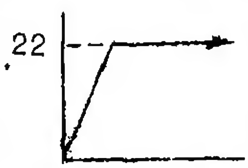
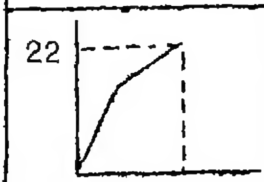
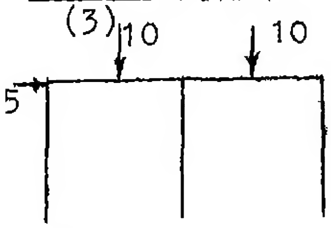
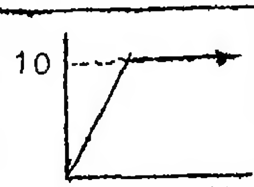
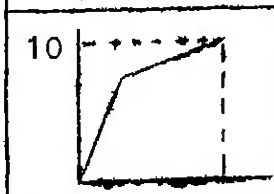
The solution of this problem is tabulated below in Table 6.7.

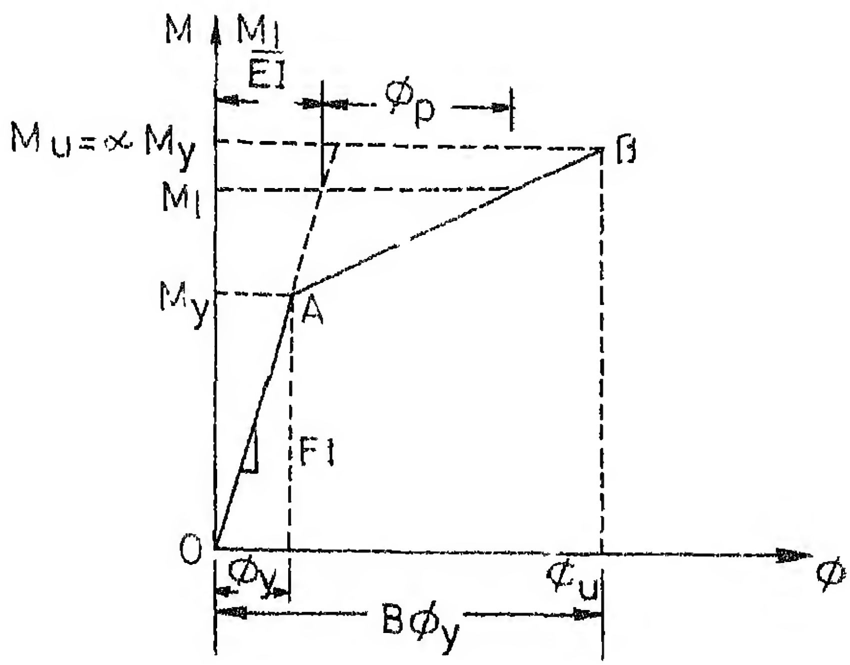
TABLE 6.7

Section	1	2	3	4	5	6	7	8	9	10
M	-3.61	-0.82	10.00	-10.0	-6.0	-4.0	10.0	6.83	8.39	6.80
$\frac{6EI\theta_i}{L}$	0.0	0.0	3.89	4.65	0.0	0.0	1.38	0.0	0.0	0.0
$\frac{6EI\bar{\theta}_{pi}}{L}$	0.0	0.0	12.78	4.65	0.0	0.0	12.16	0.0	0.0	0.0
Load factor = 1.54										

Reduction in load factor due to limited ductility is 16.3 percent.

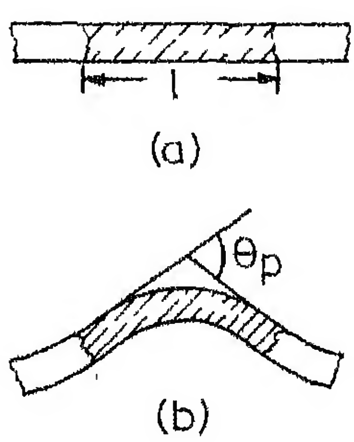
The following table gives a comparision of the ultimate load calculated with and without limited ductility for a few typical structures.

EXAMPLE	M-Ø Diagram	ULTIMATE LOAD FACTOR	REDUCTION
(1)  $M_u = \pm 22.0$ $\alpha = 1.1$ $B = 6.1$		3.3	15%
		2.8	
(2)  $M_u = \pm 22.0$ $\alpha = 1.1$ $B = 3.6$		2.64	12.2%
		2.32	
(3)  $M_u = \pm 10.0$ $\alpha = 1.1$ $B = 6.0$		1.83	16.3%
		1.54	



General Bilinear M-ϕ Curve

Figure 6 1



Inelastic Length

Figure 6-2

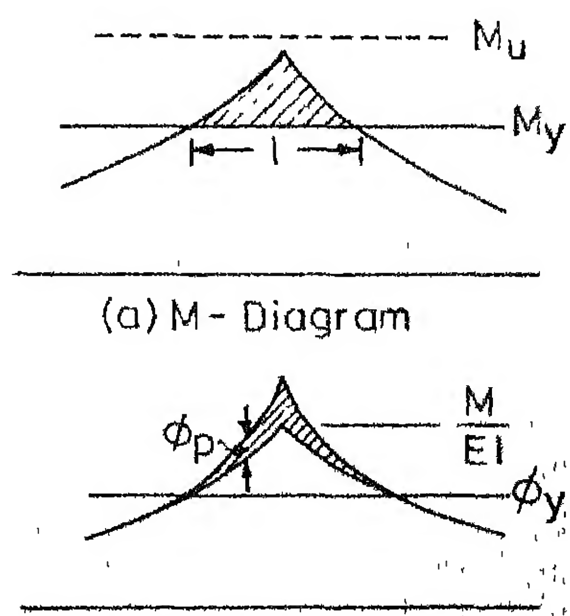
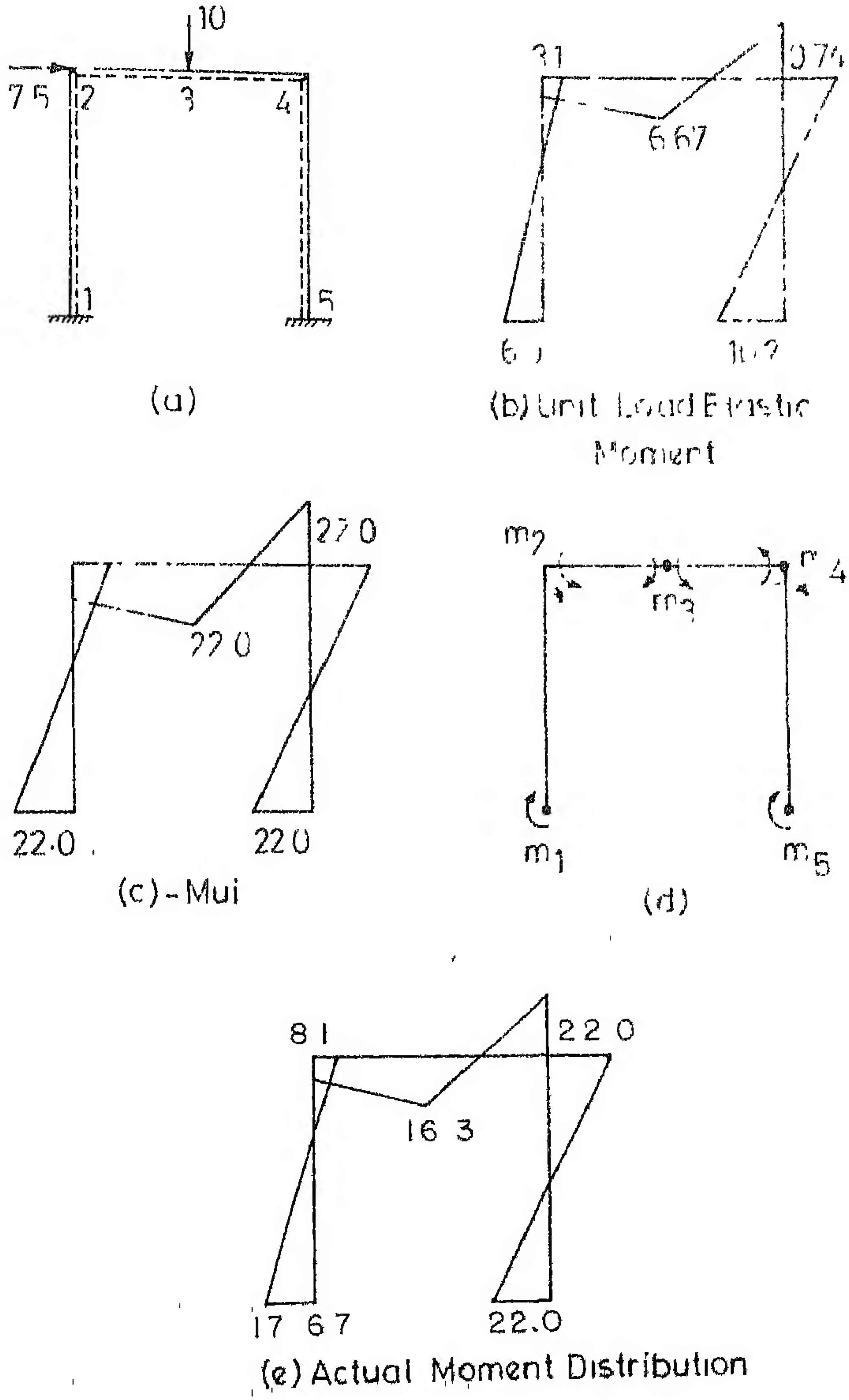
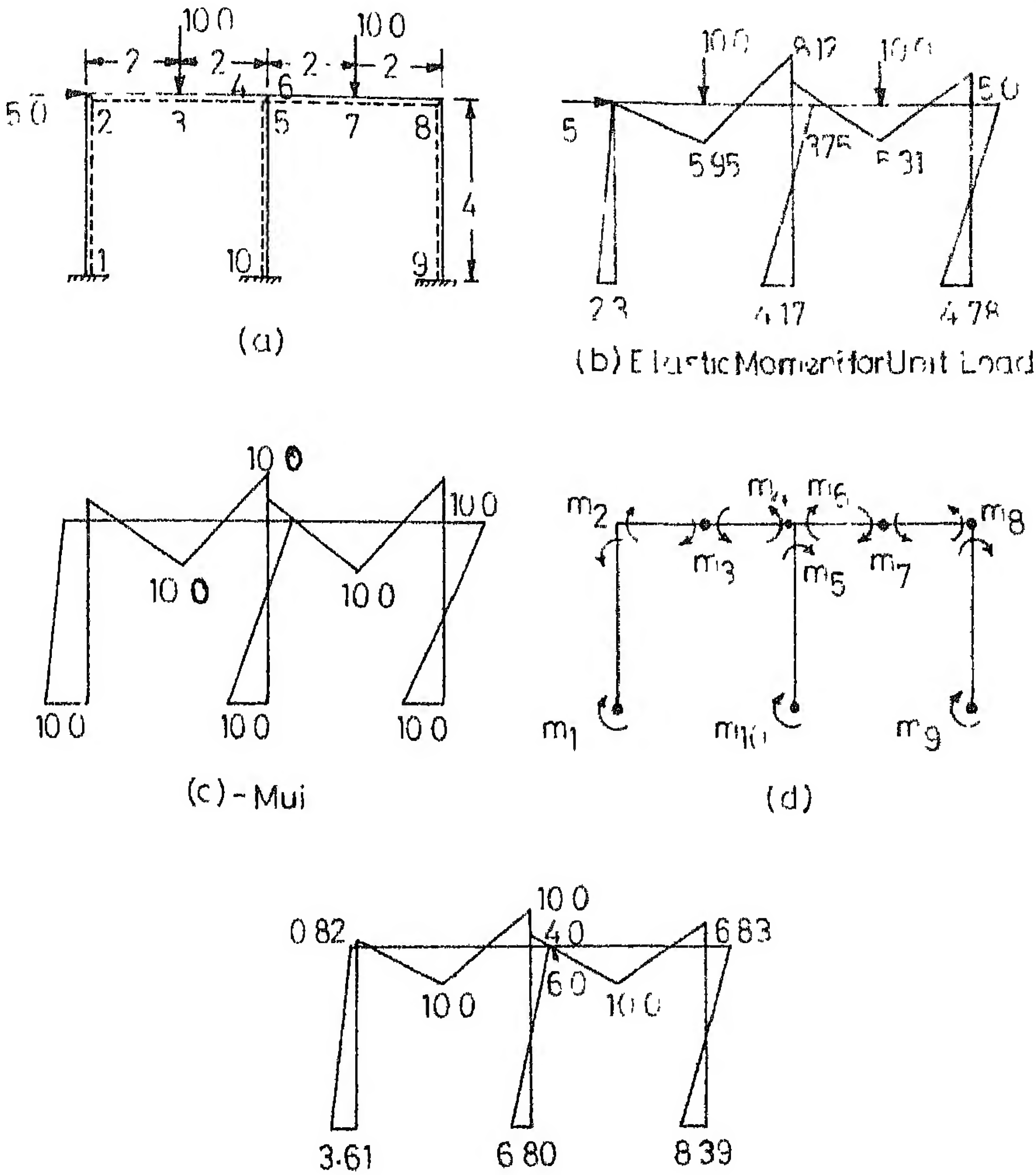


Figure 6.3



Portal Frame of Example 6.2

Figure 6.4



(e) Actual Moment Distribution.

Portal Frame of Example 6·3.

Figure 6-5

CHAPTER VII

CONCLUSIONS

1. Proposed method using linear programming can be very effectively used to arrive at a unique solution in plastic analysis of structures satisfying equilibrium, yield and mechanism conditions.

2. For analysis or design of frames larger than about 3 to 4 indeterminacy, one has to always resort to computer technique. The proposed method is ideally suited to computer.

3. Strain hardening effect is a very important factor to be taken into account for rational evaluation of limit loads. The proposed method very conveniently takes this into account.

4. For materials with limited ductility, rotation compatibility has to be satisfied. This also is convenien-tly incorporated into proposed method.

5. From the examples worked out, it is seen that for a structure with limited ductility, the ultimate load is reduced upto the extent of 15 - 20%. However, this extent of reduction depends on available ductility.

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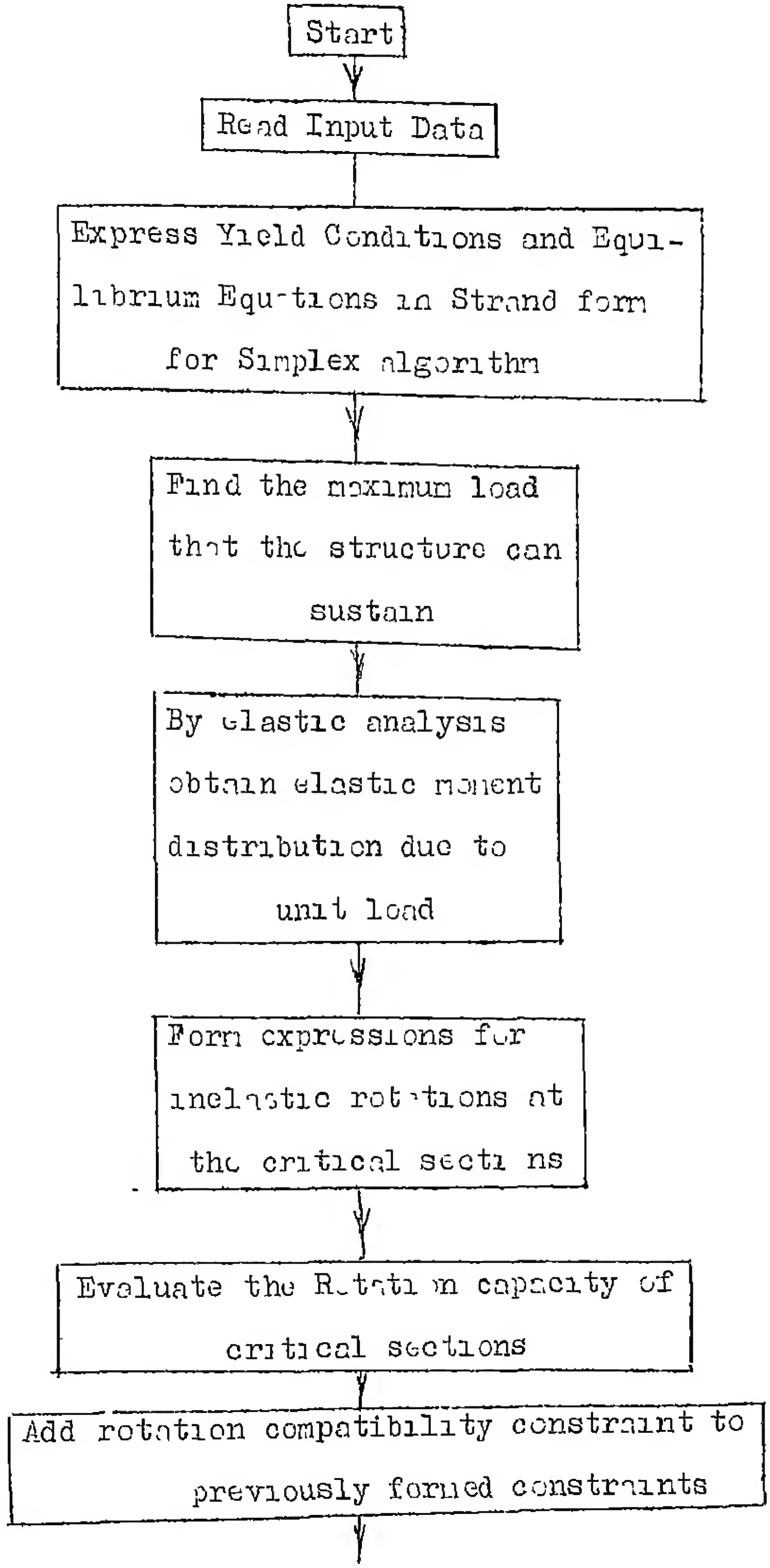
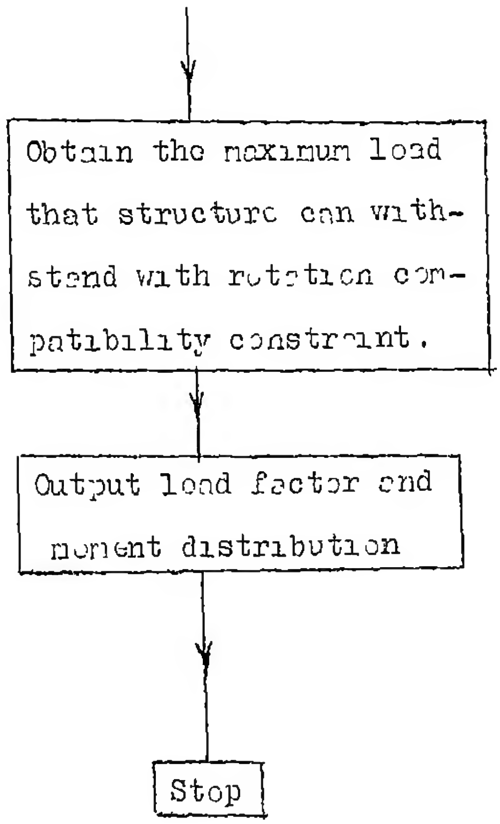
APPENDIX I

FLOW CHART FOR COMPUTER PROGRAMMING




Obtain the maximum load that structure can with-stand with rotation com-patibility constraint.
Output load factor and moment distribution
Stop

ERRATA

Page	Line	Read	For
4	19	distribution is	distribution are
9	13	beam mechanism H	beam mechanism P
9	14	panel mechanism P	panel mechanism H
10	9	indeterminacies	indeterminacies rotations
25	7	shown in Fig.7	shown in Fig.3.1
32	20	imposed at k	imposed at K
35	15	sign, m_1	sign m_1
36	17	Eq.(4.7)	Eq.(6.7)
41	8	$X_j \leq M^+_{pj} - M^-_{pj}$	$X_j \leq M^+_{pj} + M^-_{pj}$
41	14	$M^+_{pj} - M^-_{pj}$	$M^+_{pj} + M^-_{pj}$
42	11	$M_j = M^+_{pj}$	$M_j = M^+_{pi}$
42	16	$M^+_{pj} - M^-_{pj}$	$M^+_{pj} - M_p$
42	18	$M_j = M^-_{pj}$	$M_j = -M^-_{pj}$
46	13	two columns	two sections
52	7	$, \alpha , B , EI$	$, \quad , B , EI$
52	12	inelastic length	hinge length
52	20	inelastic length	hinge length

Sinha,
Analysis of inelastic frames with strain hardening and limited ductility.

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